6

MATHS SUMMER WORK

PASSPORT TO POST-16

Section 1

WATCH OR READ ONLY

In section 1 we want you to either read the examples & solutions OR click on the laptop image to watch a quick YouTube clip on the topic. This section has 7 parts to it and we only want you to WATCH at this point.

1.1 Fractions

To add or subtract fractions, find the lowest common denominator of the two fractions and then rewrite the fractions accordingly. Ensure that you simplify as far as possible.

Examples

$$\frac{2}{3}+\frac{3}{4}=\frac{8}{12}+\frac{9}{12}$$
$$=\frac{17}{12}$$
$$=1\frac{5}{12}$$

$$\frac{6}{7}-\frac{3}{5}=\frac{30}{35}-\frac{21}{35}$$
$$=\frac{9}{35}$$

When multiplying fractions, it is far more efficient to "cancel" first; this avoids trying to simplify fractions with unnecessarily large numerators and/or denominators.
To multiply with fractions, simply multiply the numerators and denominators together.

Example

$$\frac{2}{9}\times\frac{6}{7}=\frac{2}{3}\times\frac{2}{7}$$
$$=\frac{4}{21}$$

To divide by a fraction, we simply multiply by the reciprocal of the second fraction (i.e. we "flip the second fraction over").

Example

$$\frac{2}{3}\div\frac{10}{9}=\frac{2}{3}\times\frac{9}{10}$$
$$=\frac{1}{1}\times\frac{3}{5}$$
$$=\frac{3}{5}$$

For addition and subtraction with mixed numbers, add (or subtract) the integer (whole number) parts first and then work with the fractions.

Examples

$$\begin{aligned} 2\frac{1}{4}+3\frac{1}{2} &= 5\frac{1}{4}+\frac{1}{2} \\ &= 5\frac{1}{4}+\frac{2}{4} \\ &= 5\frac{1}{4}+\frac{2}{4} \\ &= 5\frac{3}{4} \end{aligned}$$

$$\begin{aligned} 3\frac{1}{5}-1\frac{2}{3} &= 2\frac{1}{5}-\frac{2}{3} \\ &= 2\frac{3}{15}-\frac{10}{15} \\ &= 2\frac{-7}{15} \\ &= 2-\frac{7}{15} \\ &= 1\frac{8}{15} \end{aligned}$$

To multiply and divide with mixed, convert the mixed numbers to improper fractions and then calculate as normal.

$$\begin{aligned} 2\frac{1}{4}\times 3\frac{1}{5} &= \frac{9}{4}\times\frac{16}{5} \\ &= \frac{9}{1}\times\frac{4}{5} \\ &= \frac{36}{5} \\ &= 7\frac{1}{5} \end{aligned}$$

$$\begin{aligned} 5\frac{1}{4}\div 2\frac{1}{2} &= \frac{21}{4}\div\frac{5}{2} \\ &= \frac{21}{4}\times\frac{2}{5} \\ &= \frac{21}{2}\times\frac{1}{5} \\ &= \frac{21}{10} \\ &= 2\frac{1}{10} \end{aligned}$$

It should also be noted that in the study of A Level Mathematics, answers are preferred as improper fractions rather than mixed numbers.

1.2 Expanding Brackets

To remove a single bracket multiply every term in the bracket by the number or expression outside:

Examples

1) $3(x + 2y) = 3x + 6y$

2) $-2(2x - 3) = (-2)(2x) + (-2)(-3)$
$= -4x + 6$

To expand two brackets multiply everything in the first bracket by everything in the second bracket. You may have used

* the smiley face method
* FOIL (First Outside Inside Last)
* using a grid.

Examples:

1) $(x + 1)(x + 2) = x(x + 2) + 1(x + 2)$

or

$(x + 1)(x + 2) = x^2 + 2 + 2x + x$
$= x^2 + 3x + 2$

or

	x	1
x	x^2	x
2	$2x$	2

$(x + 1)(x + 2) = x^2 + 2x + x + 2$
$= x^2 + 3x + 2$

2) $(x - 2)(2x + 3) = x(2x + 3) - 2(2x + 3)$
$= 2x^2 + 3x - 4x - 6$
$= 2x^2 - x - 6$

or

$(x - 2)(2x + 3) = 2x^2 - 6 + 3x - 4x = 2x^2 - x - 6$

or

	x	-2
$2x$	$2x^2$	$-4x$
3	$3x$	-6

$(2x + 3)(x - 2) = 2x^2 + 3x - 4x - 6$
$= 2x^2 - x - 6$

1.3 *Linear Equations*

When solving an equation whatever you do to one side must also be done to the other.
You may

- add the same amount to both side
- subtract the same amount from each side
- multiply the whole of each side by the same amount
- divide the whole of each side by the same amount.

If the equation has unknowns on both sides, collect all the letters onto the same side of the equation.

If the equation contains brackets, you often start by expanding the brackets.

A linear equation contains only numbers and terms in *x*.

Example 1: Solve the equation $64 - 3x = 25$

Solution: There are various ways to solve this equation. One approach is as follows:

Step 1: Add 3*x* to both sides (so that the *x* term is positive): $64 = 3 + 25$

Step 2: Subtract 25 from both sides: $39 = 3x$

Step 3: Divide both sides by 3: $13 = x$

So the solution is *x* = 13.

Example 2: Solve the equation $6 + 7 = 5 - 2x$.

Solution:
Step 1: Begin by adding 2*x* to both sides $8 + 7 = 5$
(to ensure that the *x* terms are together on the same side)

Step 2: Subtract 7 from each side: $8x = -2$

Step 3: Divide each side by 8: $x = -\frac{1}{4}$

1.4 Linear Inequalities

Linear inequalities can be solved using the same techniques as linear equations (for the most part). We may add and subtract the same numbers on both sides and we can also multiply and divide by positive numbers; multiplying/dividing both sides by a negative needs further explanation.

Example

$$2x - 3 < 11$$

Here we can simply add 3 to both sides: $2x < 14$

Next, as with linear equations we divide by 2: $x < 7$

However, if we were to have $3 – 2x > 6$, we would need to adopt a different technique. If we wish to divide or multiply by a negative number, we must reverse the direction of the inequality.

Example

$$3 - 2 > 6$$

As before, we would subtract 3 from both sides: $-2 > 3$

Divide by -2 and subsequently reverse the inequality: $< -\frac{3}{2}$

We can see this working on a more basic level; it is true to state that $3 < 4$ but it is incorrect if we multiply both sides by a negative and keep the sign as it was: $-6 < -8$ is not true.

You may find it easier to rearrange the inequality:

Example

$$3 - 2x > 6$$

If we add $2x$ to both sides, we remove the hassle: $3 > 6 + 2x$

We then subtract 6: $-3 > 2x$

Divide by two as normal: $-\frac{3}{2} > x$

Remember that you can change this round to say $< -\frac{3}{2}$

Both of these techniques are acceptable and is more a matter of preference.

1.5 Factorising Linear Expressions

Example 1: Factorise $12x – 30$

Solution: 6 is a common factor to both 12 and 30. Factorise by taking 6 outside a bracket:

$$12x - 30 = 6(2x - 5)$$

Example 2: Factorise $6x^2 – 2xy$

Solution: 2 is a common factor to both 6 and 2. Both terms also contain an *x*. Factorise by taking $2x$ outside a bracket.

$$6x^2 - 2xy = 2x(3x - y)$$

Example 3: Factorise $9x^3y^2 - 18x^2y$

Solution: 9 is a common factor to both 9 and 18.

The highest power of x that is present in both expressions is x^2.

There is also a y present in both parts.

So we factorise by taking $9x^2y$ outside a bracket:

$$9x^3y^2 - 18x^2y = 9x^2y(xy - 2)$$

Example 4: Factorise $3x(2x - 1) - 4(2x - 1)$

Solution: There is a common bracket as a factor.

So we factorise by taking $(2x - 1)$ out as a factor.

The expression factorises to $(2x - 1)(3x - 4)$

1.6 Factorising Linear Expressions

Example 1: Factorise $12x - 30$

Solution: 6 is a common factor to both 12 and 30. Factorise by taking 6 outside a bracket:

$$12x - 30 = 6(2x - 5)$$

Example 2: Factorise $6x^2 - 2xy$

Solution: 2 is a common factor to both 6 and 2. Both terms also contain an x.
Factorise by taking $2x$ outside a bracket.

$$6x^2 - 2xy = 2x(3x - y)$$

Example 3: Factorise $9x^3y^2 - 18x^2y$

Solution: 9 is a common factor to both 9 and 18.
The highest power of x that is present in both expressions is x^2.
There is also a y present in both parts.
So we factorise by taking $9x^2y$ outside a bracket:

$$9x^3y^2 - 18x^2y = 9x^2y(xy - 2)$$

Example 4: Factorise $3x(2x - 1) - 4(2x - 1)$

Solution: There is a common bracket as a factor.
So we factorise by taking $(2x - 1)$ out as a factor.
The expression factorises to $(2x - 1)(3x - 4)$

1.7 Function Notation

In GCSE Mathematics equations are written as shown below:

$y = 3x + 4$ $\qquad\qquad$ $y = x^2 + 5$

However, we also used **function notation**.

We often use the letters f and g and we write the above equations as

$f(x) = 3x + 4$ $\qquad\qquad$ $g(x) = x^2 + 5$

Example 1:

Using the equation $y = 3x + 4$, find the value of y if

(a) $x = 4$ $\qquad\qquad$ (b) $x = -6$

(a) $y = 3(4) + 4 = 12 + 4 = 16$ ⟵ Substitute for $x = 4$ in the equation

(b) $y = 3(-6) + 4 = -18 + 4 = -14$ ⟵ Substitute for $x = -6$ in the equation

Example 2:

f is a function such that $f(x) = 3x + 4$

Find the values of

(a) $f(4)$ (b) $f(-6)$

(a) $f(4) = 3(4) + 4 = 12 + 4 = 16$ ⟵ Substitute for $x = 4$ in the equation

(b) $f(-6) = 3(-6) + 4 = -18 + 4 = -14$ ⟵ Substitute for $x = -6$ in the equation

Example 3:

g is a function such that $g(x) = 2x^2 - 5$

Find the values of (a) $g(3)$ (b) $g(-4)$

(a) $g(3) = 2(3)^2 - 5 = 18 - 5 = 13$ ⟵ Substitute for $x = 3$ in the equation

(b) $g(-4) = 2(-4)^2 - 5 = 32 - 5 = 27$ ⟵ Substitute for $x = -4$ in the equation

Example 4:

The functions f and g are defined for all real values of x and are such that

$f(x) = x^2 - 4$ and $g(x) = 4x + 1$

Find (a) $f(-3)$ (b) $g(0.3)$

(c) Find the two values of x for which $f(x) = g(x)$.

(a) $f(-3) = (-3)^2 - 4 = 9 - 4 = 5$ ⟵ Substitute for $x = -3$ in the equation $f(x)$

(b) $g(0.3) = 4(0.3) + 1 = 1.2 + 1 = 2.2$ ⟵ Substitute for $x = 0.3$ in the equation $g(x)$

(c) $x^2 - 4 = 4x + 1$ ⟵ Put $f(x) = g(x)$

$x^2 - 4 - 4x - 1 = 0$ ⟵ Rearrange the equation as a quadratic = 0

$x^2 - 4x - 5 = 0$ ⟵ Simplify

$(x - 5)(x + 1) = 0$ ⟵ Solve the quadratic by factorising

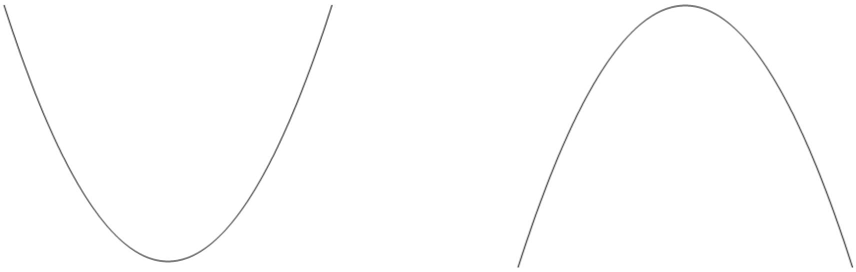
$x - 5 = 0$ or $x + 1 = 0$

$x = 5$ or $x = -1$

1.8 *Basic shapes of curved graphs*

You need to know the names of standard types of expressions, and the graphs associated with them.

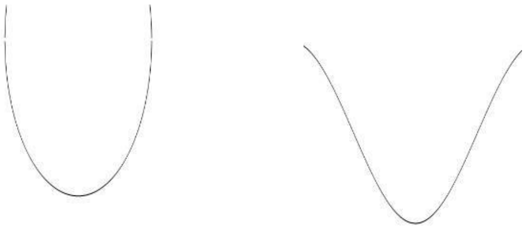
(a) The graph of a **quadratic** function (e.g. $y = 2x^2 + 3x + 4$) is a ***parabola:***



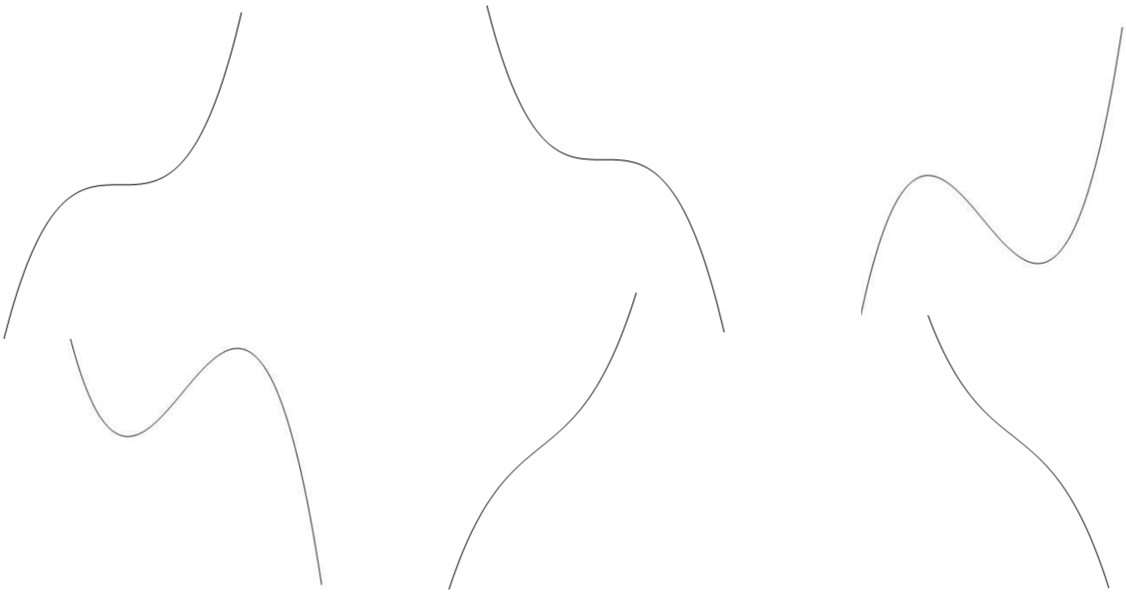
Notes:

- Parabolas are symmetric about a vertical line.
- They are not U-shaped, so the sides never reach the vertical. Neither do they dip outwards at the ends.

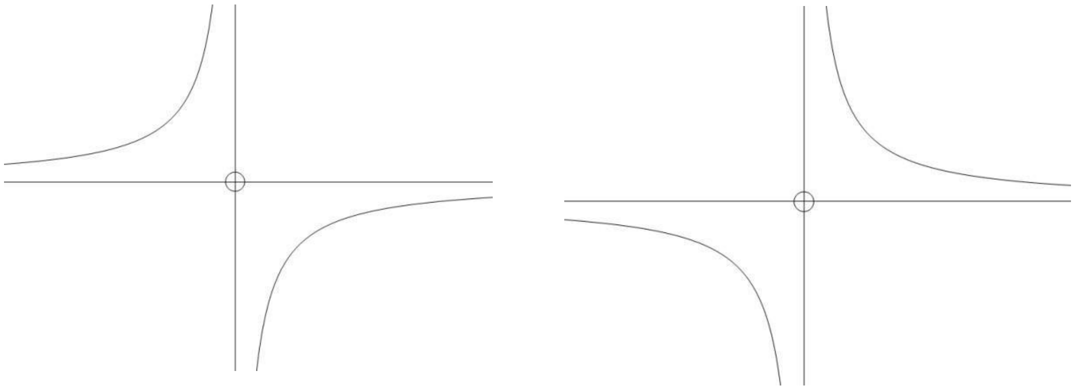
These are wrong:



(b) The graph of a **cubic** function (e.g. $y = 2x^3 – 3x^2 + 4x – 5$) has no particular name; it's usually referred to simply as a **cubic graph**. It can take several possibleshapes:

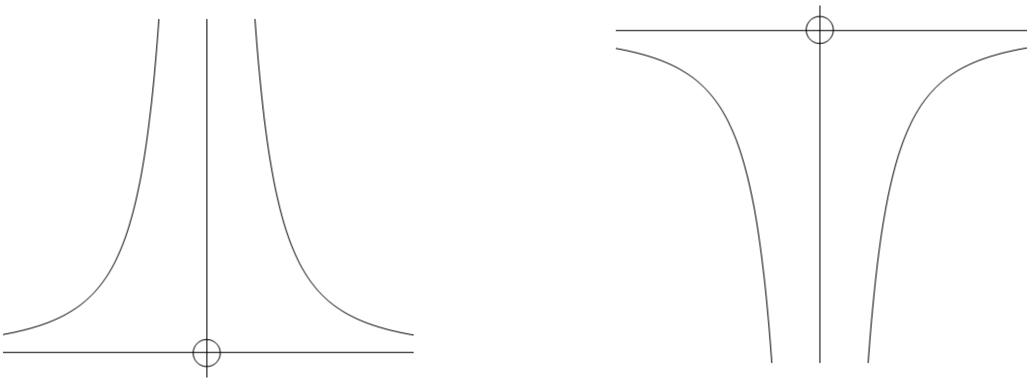


(c) The graph of $y = \frac{\text{a number}}{x}$ is a ***hyperbola***:

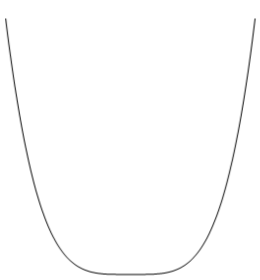


The graph of a hyperbola gets closer and closer to the axes without ever actually touching them. This is called **asymptotic** behaviour, and the axes are referred to as the **asymptotes** of this graph.

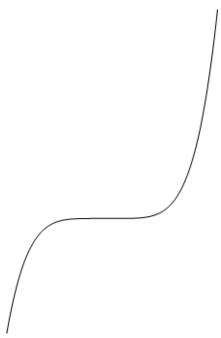
(d) The graph of $y = \frac{\text{a number}}{x^2}$ is similar (but not identical) to a hyperbola to the right but is in a different quadrant to the left:



(e) Graphs of higher *even* powers $y = x^4$ ($y = x^6$ etc. are similar):

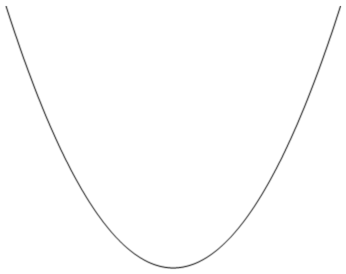


(f) Graphs of higher *odd* powers $y = x^5$ ($y = x^7$ etc. are similar):

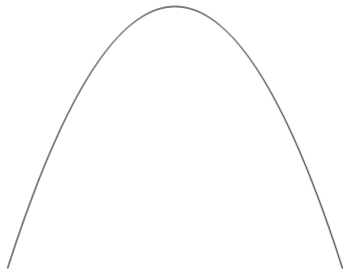


Which way up? This is determined by the *sign of the highest power*.
If the sign is positive, the *right-hand* side is (eventually) *above the x-axis*.
This is because for big values of x the highest power dominates the expression.
(If $x = 1000$, x^3 is bigger than $50x^2$).

Examples $y = x^2 - 3x - 1$

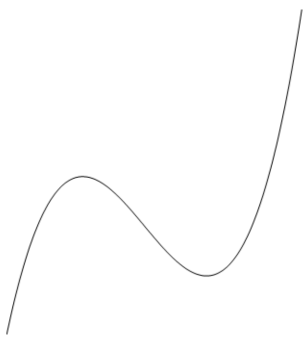


$y = 10 - x^2$

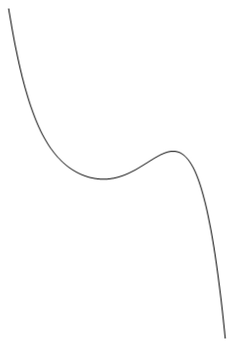


These are often referred to (informally!) as ***happy*** and ***sad*** parabolas respectively ☺ □ .

$y = x^3 - 3x - 2$



$y = 2 - x - x^5$



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Well done! You have reached the end of Section 1.

If anything has troubled you on these 8 topics please let us know via email. Mr Steinhobel is contactable on f.steinhobel@tuptonhall.org.uk

If anything was strange, challenging or new to you on Section 1, we want you to consider Core Maths instead of Alevel Maths. Please access the Core Maths Passport to Post 16 work instead, and let us know you are doing that.

From experience, students that find section 1 challenging or rusty normally find A-level Maths unbearable. If section 1 was merely like watching a brief refresher of something you already know really well, please continue with Section 2 (The Doing Section).

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Section 2
The Doing Section

We want you to practice the 19 skills in this section.

1) Click on the laptop icon and watch a short teaching clip OR read the example notes and solution
2) **COMPLETE AT LEAST HALF OF THE QUESTIONS IN THE EXERCISE** (no need to do more than half of the exercise if you are fluent in the skill. Answers are in the back of the booklet: **MARK YOUR OWN WORK.**

2.1 Equations Containing Fractions

When an equation contains a fraction, the first step is usually to multiply through by the denominator of the fraction. This ensures that there are no fractions in the equation.

Example 4 Solve the equation $y + 10 = 22$

Step 1: Multiply through by 2 (the denominator in the fraction): $y + 10 = 22$

Step 2: Subtract 10: $y = 12$

Example 5: Solve the equation $\frac{1}{3}(2x + 1) = 5$

Solution:

Step 1: Multiply by 3 (to remove the fraction) $2x + 1 = 15$

Step 2: Subtract 1 from each side $2x = 14$

Step 3: Divide by 2 $x = 7$

When an equation contains two fractions, you need to multiply by the lowest common denominator. This will then remove both fractions.

Example 6: Solve the equation $\frac{x+1}{4} + \frac{x+2}{5} = 2$

Solution:

Step 1: Find the lowest common denominator: The smallest number that both 4 and 5 divide into is 20.

Step 2: Multiply both sides by the lowest common denominator $\frac{20(x+1)}{4} + \frac{20(x+2)}{5} = 40$

Step 3: Simplify the left hand side: $\frac{\overset{5}{\cancel{20}}(x+1)}{\cancel{4}} + \frac{\overset{4}{\cancel{20}}(x+2)}{\cancel{5}} = 40$

$5(x + 1) + 4(x + 2) = 40$

Step 4: Multiply out the brackets: $5x + 5 + 4x + 8 = 40$

Step 5: Simplify the equation: $9x + 13 = 40$

Step 6: Subtract 13 $9x = 27$

Step 7: Divide by 9: $x = 3$

2.2 Forming Equations

Example 8: Find three consecutive numbers so that their sum is 96.

Solution: Let the first number be n, then the second is $n + 1$ and the third is $n + 2$.

Therefore $n + (n + 1) + (n + 2) = 96$

$3n + 3 = 96$

$3n = 93$

$n = 31$ So the numbers are 31, 32 and 33.

1) Find 3 consecutive even numbers so that their sum is 108.

2) The perimeter of a rectangle is 79 cm. One side is three times the length of the other. Form an equation and hence find the length of each side.

3) Two girls have 72 photographs of celebrities between them. One gives 11 to the other and finds that she now has half the number her friend has.
   Form an equation, letting n be the number of photographs one girl had at the **beginning**.
   Hence find how many each has **now**.

2.3 Simultaneous Equations

Example

$3x + 2y = 8$ [1]

$5x + y = 11$ [2]

x and y stand for two numbers. Solve these equations in order to find the values of x and y by eliminating one of the letters from the equations.

In these equations it is simplest to eliminate y. Make the coefficients of y the same in both equations. To do this multiply equation □ by 2, so that both equations contain $2y$:

$3x + 2y = 8$ [1]

$10x + 2y = 22$ 2×[2] = [3]

To eliminate the y terms, subtract equation [3] from equation [1]. We get: $7x = 14$

i.e. $x = 2$

To find y substitute $x = 2$ into one of the original equations. For example put it into [2]:

$10 + y = 11$

$y = 1$

Therefore the solution is $x = 2$, $y = 1$.

Remember: <u>Check</u> your solutions by substituting both x and y into the original equations.

Example: Solve $2x + 5y = 16$ [1]

$3x – 4y = 1$ [2]

Solution: Begin by getting the same number of x or y appearing in both equation. Multiply the top equation by 4 and the bottom equation by 5 to get $20y$ in both equations:

$8x + 20y = 64$ [3]

$15x – 20y = 5$ [4]

As the SIGNS in front of 20y are DIFFERENT, eliminate the y terms from the equations by ADDING:

$23x = 69$ [3]+[4]

i.e. $x = 3$

Substituting this into equation [1] gives:

$6 + 5y = 16$

$5y = 10$

So… $y = 2$

The solution is $x = 3$, $y = 2$.

Exercise: Solve the pairs of simultaneous equations in the following questions:

1) $x + 2y = 7$
$3x + 2y = 9$

2) $x + 3y = 0$
$3x + 2y = -7$

3) $3x – 2y = 4$
$2x + 3y = -6$

4) $9x – 2y = 25$
$4x – 5y = 7$

5) $4a + 3b = 22$
$5a – 4b = 43$

6) $3p + 3q = 15$
$2p + 5q = 14$

2.4 Factorising Quadratic Expressions

Simple quadratics: Factorising quadratics of the form $\boldsymbol{x + bx + c}$

The method is:

Step 1: Form two brackets (x …)(x …)

Step 2: Find two numbers that multiply to give c and add to make b. Write these two numbers at the end of the brackets.

Example 1: Factorise $x^2 – 9x – 10$.

Solution: Find two numbers that multiply to make -10 and add to make -9. These numbers are -10 and 1.

Therefore $x^2 – 9x – 10 = (x – 10)(x + 1)$.

General quadratics: Factorising quadratics of the form $\boldsymbol{ax + bx + c}$

One method is that of combining factors. There are many more options that you can use.

Another method is:

Step 1: Find two numbers that multiply together to make ac and add to make b.

Step 2: Split up the bx term using the numbers found in step 1.

Step 3: Factorise the front and back pair of expressions as fully as possible.

Step 4: There should be a common bracket. Take this out as a common factor.

Example 2: Factorise $6x^2 + x - 12$.

Solution: We need to find two numbers that multiply to make $6 \times -12 = -72$ and add to make 1. These two numbers are -8 and 9.

Therefore, $6x^2 + x - 12 = \underbrace{6x^2 - 8x}\ \underbrace{+ 9x - 12}$

$= 2x(3x - 4) + 3(3x - 4)$ (the two brackets must be identical)

$= (3x - 4)(2x + 3)$

Difference of two squares: Factorising quadratics of the form $x^2 - a^2$

Remember that $x^2 - a^2 = (x + (a)(x - (a)$.

Therefore: $x^2 - 9 = x^2 - 3^2 = (x + 3)(x - 3)$

$16x^2 - 25 = (2x)^2 - 5^2 = (4x + 5)(4x - 5)$

Also notice that: $2x^2 - 8 = 2(x^2 - 4) = 2(x + 4)(x - 4)$

and $3x^3 - 48xy^2 = 3x(x^2 - 16y^2) = 3x(x + 4y)(x - 4y)$

Exercise B:

Factorise

1) $x^2 - x - 6$

2) $x^2 + 6x - 16$

3) $2x^2 + 5x + 2$

4) $2x^2 - 3x$

5) $3x^2 + 5x - 2$

6) $2y^2 + 17y + 21$

7) $7y^2 - 10y + 3$

8) $10x^2 + 5x - 30$

9) $4x^2 - 25$

10) $x^2 - 3x - xy + 3y$

11) $4x^2 - 12x + 8$

12) $16m^2 - 81n^2$

13) $4y^3 - 9a^2y$

14) $8(x + 1)^2 - 2(x + 1) - 10$

EXERCISE B Factorise

1) $x^2 - x - 6$

2) $x^2 + 6x - 16$

3) $2x^2 + 5x + 2$

4) $2x^2 - 3x$

5) $3x^2 + 5x - 2$

6) $2y^2 + 17y + 21$

7) $7y^2 - 10y + 3$

8) $10x^2 + 5x - 30$

9) $4x^2 - 25$

10) $x^2 - 3x - xy + 3y$

11) $4x^2 - 12x + 8$

12) $16m^2 - 81n^2$

13) $4y^3 - 9a^2y$

14) $8(x + 1)^2 - 2(x + 1) - 10$

2.5 Completing the Square

A related process is to write a quadratic expression such as $x^2 + 6x + 11$ in the form $(x + a)^2 + b$. This is called ***completing the square***. It is often useful, because $x^2 + 6x + 11$ is not a very transparent expression – it contains x in more than one place, and it's not easy either to rearrange or to relate its graph to that of x^2.

Completing the square for quadratic expressions in which the coefficient of x^2 is 1 (these are called ***monic quadratics***) is very easy. The number a inside the brackets is always half of the coefficient of x.

Example 1 Write $x^2 + 6x + 4$ in the form $(x + a)^2 + b$.

Solution $x^2 + 6x + 4$ is a monic quadratic, so a is half of 6, namely 3.
When you multiply out $(x + 3)^2$, you get $x^2 + 6x + 9$.
[The x-term is always twice a, which is why you have to halve it to get a.]
$x^2 + 6x + 9$ isn't quite right yet; we need 4 at the end, not 9, so we can write

$$x^2 + 6x + 4 = (x + 3)^2 - 9 + 4$$
$$= (x + 3)^2 - 5.$$

This version immediately gives us several useful pieces of information. For instance, we now know a lot about the graph of $y = x^2 + 6x + 4$:

- It is a translation of the graph of $y = x^2$ by 3 units to the left and 5 units down
- Its line of symmetry is $x = -3$
- Its lowest point or vertex is at $(-3, -5)$

We also know that the smallest value of the function $x^2 + 6x + 4$ is –5 and this occurs when $x = -3$.

And we can solve the equation $x^2 + 6x + 4 = 0$ *exactly* without having to use the quadratic equation formula, to locate the roots of the function:

$$x^2 + 6x + 4 = 0$$
$$\Rightarrow \quad (x + 3)^2 - 5 = 0$$
$$\Rightarrow \quad (x + 3)^2 = 5$$
$$\Rightarrow \quad (x + 3) = \pm\sqrt{5} \qquad \text{[don't forget that there are two possibilities!]}$$
$$\Rightarrow \quad x = -3 \pm \sqrt{5}$$

These are of course the same solutions that would be obtained from the quadratic equation formula – not very surprisingly, as the formula itself is obtained by completing the square for the general quadratic equation $ax^2 + bx + c = 0$.

Non-monic quadratics

Everyone knows that non-monic quadratic expressions are hard to deal with. Nobody really likes trying to factorise $6x^2 + 5x - 6$ (although you should certainly be willing and able to do so for A Level, which is why some examples are included in the exercises here).

Example 2 Write $2x^2 + 12x + 23$ in the form $a(x + b)^2 + c$.

Solution First take out the factor of 2:

$$2x^2 + 12x + 23 = 2(x^2 + 6x + 11.5) \qquad \text{[you can ignore the 11.5 for now]}$$

Now we can use the method for monic quadratics to write

$$x^2 + 6x + 11.5 = (x + 3)^2 + (\text{something})$$

Half of 6

So we have, so far

$$2x^2 + 12x + 23 = 2(x + 3)^2 + c \qquad \text{[so we already have } a = 2 \text{ and } b = 3\text{]}$$

Now $2(x + 3)^2 = 2(x^2 + 6x + 9)$
$$= 2x^2 + 12x + 18$$

We want 23 at the end, not 18, so:

$$2x^2 + 12x + 23 = 2(x + 3)^2 - 18 + 23$$
$$= 2(x + 3)^2 + 5.$$

If the coefficient of x^2 is a perfect square you can sometimes get a more useful form.

Example 3 Write $4x^2 + 20x + 19$ in the form $(ax + b)^2 + c$.

Solution It should be obvious that $a = 2$ (the coefficient of a^2 is 4).

So $4x^2 + 20x + 19 = (2x + b)^2 + c$

If you multiply out the bracket now, the middle term will be $2 \times 2x \times b = 4bx$.

So $4bx$ must equal $20x$ and clearly $b = 5$.

And we know that $(2x + 5)^2 = 4x^2 + 20x + 25$.

So $$4x^2 + 20x + 19 = (2x + 5)^2 - 25 + 19$$
$$= (2x + 5)^2 - 6.$$

EXERCISE A

1 Write the following in the form $(x + a)^2 + b$.

(a) $x^2 + 8x + 19$ (b) $x^2 - 10x + 23$ (c) $x^2 + 2x - 4$

(d) $x^2 - 4x - 3$ (e) $x^2 - 3x + 2$ (f) $x^2 - 5x - 6$

2 Write the following in the form $a(x + b)^2 + c$.

(a) $3x^2 + 6x + 7$ (b) $5x^2 - 20x + 17$ (c) $2x^2 + 10x + 13$

3 Write the following in the form $(ax + b)^2 + c$.

(a) $4x^2 + 12x + 14$ (b) $9x^2 - 12x - 1$ (c) $16x^2 + 40x + 22$

2.6 Solving Quadratic Equations

A quadratic equation has the form $ax^2 + bx + c = 0$.

There are two methods that are commonly used for solving quadratic equations:

* factorising
* the quadratic formula

Not all quadratic equations can be solved by factorising.

Method 1: Factorising

Make sure that the equation is rearranged so that the right hand side is 0. It usually makes it easi coefficient of x^2 is positive.

Example 1 : Solve $x^2 - 3x + 2 = 0$

Factorise $(x - 1)(x - 2) = 0$

Either $(x - 1) = 0$ or $(x - 2) = 0$

So the solutions are $x = 1$ or $x = 2$

Note: The individual values $x = 1$ and $x = 2$ are called the **roots** of the equation.

Example 2: Solve $x^2 - 2x = 0$

Factorise: $x(x - 2) = 0$

Either $x = 0$ or $(x - 2) = 0$

So $x = 0$ or $x = 2$

Method 2: Using the formula

The roots of the quadratic equation $ax^2 + bx + c = 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 3: Solve the equation $2x^2 - 5 = 7 - 3x$

Solution: First we rearrange so that the right hand side is 0. We get $2x^2 + 3x - 12 = 0$
We can then tell that $a = 2$, $b = 3$ and $c = -12$.
Substituting these into the quadratic formula gives:

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \times 2 \times (-12)}}{2 \times 2} = \frac{-3 \pm \sqrt{105}}{4}$$

(this is the *surd form* for the solutions)

If we have a calculator, we can evaluate these roots to get: $x = 1.81$ or $x = -3.31$

Exercise

1) Use factorisation to solve the following equations:

(a) $x^2 + 3x + 2 = 0$ (b) $x^2 - 3x - 4 = 0$ (c) $x^2 = 15 - 2x$

2) Find the roots of the following equations:

(a) $x^2 + 3x = 0$ (b) $x^2 - 4x = 0$ (c) $4 - x^2 = 0$

3) Solve the following equations either by factorising or by using the formula:

(a) $6x^2 - 5x - 4 = 0$ (b) $8x^2 - 24x + 10 = 0$

4) Use the formula to solve the following equations to 3 significant figures where possible

(a) $x^2 + 7x + 9 = 0$

(b) $6 + 3x = 8x^2$

(c) $4x^2 - x - 7 = 0$

(d) $x^2 - 3x + 18 = 0$

(e) $3x^2 + 4x + 4 = 0$

(f) $3x^2 = 13x - 16$

2.7 Changing the Subject

Rearranging a formula is similar to solving an equation –always do the same to both sides.

Example 1: Make x the subject of the formula $y = 4x + 3$.

Solution: $y = 4x + 3$

Subtract 3 from both sides: $y - 3 = 4x$

Divide both sides by 4; $\frac{y-3}{4} = x$

So $x = \frac{y-3}{4}$ is the same equation but with x the subject.

Example 2: Make x the subject of $y = 2 - 5x$

Solution: Notice that in this formula the x term is negative.

$y = 2 - 5x$

Add $5x$ to both sides $y + 5x = 2$ (the x term is now positive)

Subtract y from both sides $5x = 2 - y$

Divide both sides by 5 $x = \frac{2-y}{5}$

Example 3: The formula $C = \frac{5(F-32)}{9}$ is used to convert between ° Fahrenheit and ° Celsius.

Rearrange to make F the subject.

$C = \frac{5(F-32)}{9}$

Multiply by 9 $9C = 5(F-32)$ (this removes the fraction)

Expand the brackets $9C = 5F - 160$

Add 160 to both sides $9C + 160 = 5F$

Divide both sides by 5 $\frac{9C+160}{5} = F$

Therefore the required rearrangement is $F = \frac{9C+160}{5}$.

EXERCISE A Make x the subject of each of these formulae:

1) $y = 7x - 1$

2) $y = \frac{x+5}{4}$

3) $4y = \frac{x}{3} - 2$

4) $y = \frac{4(3x-5)}{9}$

Example 4: Make x the subject of $x^2 + y^2 = w^2$

Solution: $x^2 + y^2 = w^2$

Subtract y^2 from both sides: $x^2 = w^2 - y^2$ (this isolates the term involving x)

Square root both sides: $= \pm\sqrt{w^2 - y^2}$

Remember the positive & negative square root.

Example 5: Make a the subject of the formula $t = \frac{1}{4}\sqrt{\frac{5a}{h}}$

Solution: $t = \frac{1}{4}\sqrt{\frac{5a}{h}}$

Multiply by 4 $4t = \sqrt{\frac{5a}{h}}$

Square both sides $16t^2 = \frac{5a}{h}$

Multiply by h: $16t^2 h = 5a$

Divide by 5: $\frac{16t^2 h}{5} = a$

EXERCISE B: Make t the subject of each of the following

1) $P = \frac{wt}{32r}$

3) $V = \frac{1}{3}\pi t^2 h$

5) $3Pa = \frac{w(v-t)}{g}$

2) $P = \frac{wt^2}{32r}$

4) $P = \sqrt{\frac{2t}{g}}$

6) $= a + bt^2$

When the Subject Appears More Than Once

Sometimes the subject occurs in more than one place in the formula. In these questions collect the terms involving this variable on one side of the equation, and put the other terms on the opposite side.

Example 6: Make *t* the subject of the formula $a - xt = b + yt$

Solution: $a - xt = b + yt$

Start by collecting all the t terms on the right hand side:

Add xt to both sides: $= b + yt + xt$

Now put the terms without a *t* on the left hand side:

Subtract *b* from both sides: $a - = yt + xt$

Factorise the RHS: $a - = t(y + x)$

Divide by $(y + x)$: $\frac{a - b}{y + x} = t$

So the required equation is $t = \frac{a - b}{y + x}$

Example 7: Make *W* the subject of the formula $T - W = \frac{Wa}{2b}$

Solution: This formula is complicated by the fractional term. Begin by removing the fraction:

Multiply by $2b$: $2bT - 2b = Wa$

Add $2bW$ to both sides: $2bT = a + 2bW$ (this collects the W's together)

Factorise the RHS: $2bT = (a + 2b)$

Divide both sides by $a + 2b$: $W = \frac{2bT}{a + 2b}$

Exercise C Make *x* the subject of these formulae:

1) $ax + 3 = bx + c$

2) $3(x + a) = (x - 2)$

3) $y = \frac{2x + 3}{5x - 2}$

4) $\frac{x}{a} = 1 + \frac{x}{b}$

2.8 Indices

Basic rules of indices

y^4 means $y \times y \times y \times y$. 4 is called the **index** (plural: indices), **power** or **exponent** of y.

There are 3 basic rules of indices:

1) $a \times a^{n} = a^{m+n}$ e.g. $3^4 \times 3^5 = 3^9$
2) $a \div a^{n} = a^{m-n}$ e.g. $3^8 \times 3^6 = 3^2$
3) $(a) = a^{mn}$ e.g. $(3^2)^5 = 3^{10}$

Further examples

$y^4 \times 5y^3 = 5y^7$

$4a^3 \times 6a^2 = 24a^5$ (multiply the numbers and multiply the a's)

$2c^2 \times (-3c^6) = -6c^8$ (multiply the numbers and multiply the c's)

$$24d^7 \div 3d^2 = \frac{24d^7}{3d^2} = 8d^5$$

(divide the numbers and divide the d terms by subtracting the powers)

EXERCISE A Simplify the following: Remember that $= b^1$

1) $b \times 5b^5$
2) $3c^2 \times 2c^5$
3) $b^2c \times bc^3$
4) $2n^6 \times (-6n^2)$
5) $8n^8 \div 2n^3$
6) $d^{11} \div d^9$
7) $(a^3)^2$
8) $(-d^4)^3$

Zero iuclex: Remernber $a^0 = I$ For any non-zero number, *a*.

Therefore $\left(\frac{3}{4}\right)^0 = \mathbf{1}$ $(-5.2304)^0 = \mathbf{1}$

Negative powers

A power of -1 corresponds to the reciprocal of a number, i.e.

This result can be extended to more general negative powers:

This means:

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

$$\left(\frac{1}{4}\right)^{-2} = \left(\left(\frac{1}{4}\right)^{-1}\right)^2 = \left(\frac{4}{1}\right)^2 = 16$$

There is a particularly nice way of understanding the negative power .
Consider the following:

3^1	3^2	3^3	3^4	3^5
3	9	27	81	243

x3 x3 x3 x3

Eve1y time you move one step to the *right* you *mu!tip y* by 3.
ow consider the sequence continuing, right-to-left:

3^{-2}	3^{-l}	3^0	3^1	3^2	3^3	3^4	3^5
1/9	1/3	1	3	9	27	81	243

Each time you move one step to the *left* you *divide* by 3.
Take particular care when there are numbers as well as negative powers.

Fractional powers:

Click on the laptop icon and watch the short teaching clip.

EXERCISE B: Find the value of:

1) $4^{\frac{1}{2}}$

2) $27^{\frac{1}{3}}$

3) $\left(\frac{1}{9}\right)^{\frac{1}{2}}$

4) 5^{-2}

5) 18^{0}

6) 7^{-1}

7) $27^{\frac{2}{3}}$

8) $\left(\frac{2}{3}\right)^{-2}$

9) $8^{-\frac{2}{3}}$

10) $(0.04)^{\frac{1}{2}}$

11) $\left(\frac{8}{27}\right)^{2/3}$

12) $\left(\frac{1}{16}\right)^{-3/2}$

Simplify each of the following:

13) $2a^{\frac{1}{2}} \times 3a^{\frac{5}{2}}$

14) $x^{3} \times x^{-2}$

15) $(x^{2}y^{4})^{\frac{1}{2}}$

2.9 *Surds*

A surd is a root of a number that cannot be expressed as an integer. Surds are part of the set of irrational numbers.

Example:

$\sqrt{3}$ and $\sqrt{8}$ are surds but $\sqrt{4}$ is not as it equals 2.

Simplifying Surds

Start to simplify surds by using two rules: $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ and $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

By using the multiplication rule, simplify surds by finding a factor of the number you are taking a root of which is a square number. Always try to find the largest square number factor, otherwise you will have to simplify further.

Example:

$$\sqrt{8} = \sqrt{4} \times \sqrt{2}$$
$$= 2\sqrt{2}$$

$$3\sqrt{12} = 3 \times \sqrt{4} \times \sqrt{3}$$
$$= 3 \times 2 \times \sqrt{3}$$
$$= 6\sqrt{3}$$

$$\frac{\sqrt{600}}{\sqrt{2}} = \sqrt{\frac{600}{2}}$$
$$= \sqrt{300}$$
$$= \sqrt{100} \times \sqrt{3}$$
$$= 10\sqrt{3}$$

EXERCISE A

Simplify

1) $\sqrt{50}$

2) $\sqrt{72}$

3) $\sqrt{27}$

4) $\sqrt{80}$

5) $\sqrt{360}$

6) $\frac{\sqrt{900}}{\sqrt{3}}$

Multiplying and Dividing with Surds

The rules of algebra are true for any numeric value; these include surds. To multiply and divide expressions with surds, deal with any integers together and then deal with any surds.

Examples:

$$2\sqrt{3} \times \sqrt{2} = 2\sqrt{6}$$

$$3\sqrt{5} \times 6\sqrt{2} = 18\sqrt{10}$$

$$\begin{aligned} 2\sqrt{5} \times 7\sqrt{8} &= 14\sqrt{40} \\ &= 14 \times \sqrt{4} \times \sqrt{10} \\ &= 28\sqrt{10} \end{aligned}$$

$$\sqrt{2}(5 + 2\sqrt{3}) = 5\sqrt{2} + 2\sqrt{6}$$

$$\frac{8\sqrt{14}}{2\sqrt{7}} = 4\sqrt{2}$$

$$(1 + \sqrt{3})(2 - \sqrt{2}) = 2 - 2\sqrt{2} + 2\sqrt{3} - \sqrt{6}$$

$$\begin{aligned} (3 + \sqrt{2})(3 - \sqrt{2}) &= 3^2 - (\sqrt{2})^2 \\ &= 1 \end{aligned}$$

In this example, you could expand as usual but this is an example of the difference of two squares.

EXERCISE B

Simplify

1) $\sqrt{3} \times \sqrt{7}$
2) $5\sqrt{2} \times 4\sqrt{5}$
3) $3\sqrt{3} \times 2\sqrt{6}$
4) $\sqrt{8} \times \sqrt{27}$
5) $\frac{5\sqrt{20}}{6\sqrt{5}}$
6) $\frac{8\sqrt{18}}{4\sqrt{2}}$
7) $(\sqrt{2} + 1)(\sqrt{2} + 5)$
8) $(5 - \sqrt{3})(\sqrt{2} - 8)$

Addition and Subtraction with Surds

You can only add or subtract with surds if the surd is the same; sometimes if they are not the same, you may be able to simplify them so that the same surd is present.

Example:

$$2\sqrt{3} + 4\sqrt{3} + 6\sqrt{5} = 6\sqrt{3} + 6\sqrt{5}$$

Here add the $2\sqrt{3}$ and $4\sqrt{3}$ as the same surd is present but you cannot add the $6\sqrt{5}$.

$$2\sqrt{5} + \sqrt{45} = 2\sqrt{5} + 3\sqrt{5}$$
$$= 5\sqrt{5}$$

By simplifying $\sqrt{45}$ to $3\sqrt{5}$, you can add the two surds together.

These methods also work for subtraction of surds.

Exercise C

Simplify

1) $\sqrt{3} + \sqrt{7}$
2) $5\sqrt{2} + 4\sqrt{2}$
3) $3\sqrt{6} + \sqrt{24}$
4) $\sqrt{50} + \sqrt{8}$
5) $\sqrt{27} + \sqrt{75}$
6) $2\sqrt{5} - \sqrt{5}$
7) $\sqrt{72} - \sqrt{50}$
8) $6\sqrt{3} - \sqrt{12} + \sqrt{27}$
9) $\sqrt{200} + \sqrt{90} - \sqrt{98}$
10) $\sqrt{72} - \sqrt{75} + \sqrt{108}$

Rationalising the Denominator

It is far easier to calculate with a fraction if the surd if the denominator is a rational number (i.e. not a surd). The process of this is known as *rationalising the denominator*. To do this, multiply by the surd in the denominator, doing so makes use of the fact that $\sqrt{a} \times \sqrt{a} = (\sqrt{a})^2 = a$

Example:

$$\frac{1}{\sqrt{3}}$$

Multiply the denominator by $\sqrt{3}$ to rationalise it and so multiply the numerator by $\sqrt{3}$ also:

$$\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Example 2:

$$\frac{4}{\sqrt{2}} = \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{4\sqrt{2}}{2}$$
$$= 2\sqrt{2}$$

Example 3:

$$\frac{2+\sqrt{3}}{\sqrt{5}} = \frac{2+\sqrt{3}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$
$$= \frac{\sqrt{5}(2+\sqrt{3})}{5}$$
$$= \frac{2\sqrt{5}+\sqrt{15}}{5}$$

If there is more than just a surd in the denominator, we make use of the difference of two squares by multiplying by its conjugate.

Example:

Rationalise $\frac{2}{3-\sqrt{7}}$

We multiply the numerator and denominator by its conjugate: $3+\sqrt{7}$

It's a difference of two squares so expand as usual

$$\frac{2}{3-\sqrt{7}} \times \frac{3+\sqrt{7}}{3+\sqrt{7}} = \frac{2(3+\sqrt{7})}{(3-\sqrt{7})(3+\sqrt{7})}$$
$$= \frac{2(3+\sqrt{7})}{3^2-(\sqrt{7})^2}$$
$$= \frac{2(3+\sqrt{7})}{9-7}$$
$$= \frac{2(3+\sqrt{7})}{2}$$
$$= 3+\sqrt{7}$$

Exercise D

Rationalise the following:

1

a) $\frac{1}{\sqrt{2}}$ b) $\frac{3}{\sqrt{5}}$ c) $\frac{10}{\sqrt{5}}$

d) $\frac{5}{2\sqrt{7}}$ e) $\frac{\sqrt{3}}{\sqrt{2}}$ f) $\frac{10}{\sqrt{10}}$

g) $\frac{4+\sqrt{7}}{\sqrt{3}}$ h) $\frac{6+8\sqrt{5}}{\sqrt{2}}$ i) $\frac{6-\sqrt{5}}{\sqrt{5}}$

2

a) $\frac{1}{\sqrt{2}-1}$ b) $\frac{2}{\sqrt{6}-2}$ c) $\frac{6}{\sqrt{7}+2}$

d) $\frac{1}{3+\sqrt{5}}$ e) $\frac{1}{\sqrt{6}-\sqrt{5}}$

2.10 Algebraic Fractions

Algebraic fractions can be treated in exactly the same way as numerical fractions.

Example Multiply $\frac{3x}{7y}$ by 2.

Solution 3 × 2 = 6x, so the answer is $\frac{6x}{7y}$. (Not $\frac{6x}{14y}$ as this is just an equivalent fraction!)

Example Divide $\frac{3y^2}{4x}$ by y.

Solution

$$\frac{3y^2}{4x} \div y = \frac{3y^2}{4x} \times \frac{1}{y}$$
$$= \frac{3y^2}{4xy}$$
$$= \frac{3y}{4x}$$ (Don't forget to simplify!)

Example Divide $\frac{PQR}{100}$ by T.

Solution

$$\frac{PQR}{100} \div T = \frac{PQR}{100} \times \frac{1}{T}$$
$$= \frac{PQR}{100T}$$

Here it would be wrong to say just $\frac{\frac{PQR}{100}}{T}$, which is a mix (as well as a mess!)
Double fractions, or mixtures of fractions and decimals, are always wrong.

For instance, if you want to divide $\frac{xy}{z}$ by 2, you should not say $\frac{0.5xy}{z}$ but $\frac{xy}{2z}$.

This sort of thing is extremely important when it comes to rearranging formulae.

Example Simplify $\dfrac{3}{x-1} - \dfrac{1}{x+1}$

Solution Use a common denominator. [You must treat (x – 1) and (x + 1) as separate expressions with no common factor.]

$$\frac{3}{x-1} - \frac{1}{x+1} = \frac{3(x+1)-(x-1)}{(x-1)(x+1)}$$

$$= \frac{3x+3-x+1}{(x-1)(x+1)} = \frac{2x+4}{(x-1)(x+1)}.$$

Do use brackets, particularly on top – otherwise you are likely to forget the minus at the end of the numerator (in this example subtracting -1 gives +1).
Don't multiply out the brackets on the bottom. You will need to see if there is a factor, which cancels out (although there isn't one in this case).

Example Simplify $\dfrac{2}{3x-3} + \dfrac{5}{x^2-1}$.

Solution A common denominator may not be obvious, you should look to see if the denominator factorises first.

$$\frac{2}{3x-3} + \frac{5}{x^2-1} = \frac{2}{3(x-1)} + \frac{5}{(x+1)(x-1)}$$

$$= \frac{2(x+1) + 5\times 3}{3(x-1)(x+1)}$$

$$= \frac{2x+2+15}{3(x-1)(x+1)}$$

$$\frac{2x+17}{3(x-1)(x+1)}$$

> $x - 1$ is a common factor, so the common denominator is $3(x-1)(x+1)$.

If one of the terms is not a fraction already, the best plan is to make it one.

Example Write $\dfrac{3}{x+1} + 2$ as a single fraction.

Solution

$$\frac{3}{x+1} + 2 = \frac{3}{x+1} + \frac{2}{1}$$

$$= \frac{3+2(x+1)}{x+1}$$

$$= \frac{2x+5}{x+1}$$

This method often produces big simplifications when roots are involved.

Example Write $\frac{x}{\sqrt{x-2}}+\sqrt{x-2}$ as a single fraction.

Solution

$$\frac{x}{\sqrt{x-2}}+\sqrt{x-2}=\frac{x}{\sqrt{x-2}}+\frac{\sqrt{x-2}}{1}$$

$$=\frac{x+(\sqrt{x-2})^2}{\sqrt{x-2}}$$

$$=\frac{x+(x-2)}{\sqrt{x-2}}$$

$$=\frac{2x-2}{\sqrt{x-2}}$$

It is also often useful to reverse this process – that is, to rewrite expressions such as $\frac{x}{x-2}$. The problem with this expression is that x appears in more than one place and it is not very easy to manipulate such expressions (for example, in finding the inverse function, or sketching a curve). Here is a very useful trick.

Exercise

1 Write as single fractions.

(a) $\frac{2}{x-1}+\frac{1}{x+3}$ (b) $\frac{2}{x-3}-\frac{1}{x+2}$ (c) $\frac{1}{2x-1}-\frac{1}{3x+2}$ (d) $\frac{3}{x+2}+1$

(e) $2-\frac{1}{x-1}$ (f) $\frac{2x}{x+1}-3$ (g) $\frac{3}{4(2x-1)}-\frac{1}{4x^2-1}$

2 Write as single fractions.

(a) $\frac{x+1}{\sqrt{x}}+\sqrt{x}$ (b) $\frac{2x}{\sqrt{x+3}}+\sqrt{x+3}$ (c) $\frac{x}{\sqrt[3]{x-2}}+\sqrt[3]{(x-2)^2}$

2.11 Linear and Quadratic Simultaneous Equations

I am sure that you will be very familiar with the standard methods of solving simultaneous equations (elimination and substitution). You will probably have met the method for solving simultaneous equations when one equation is linear and one is quadratic. Here you have no choice; you ***must*** use substitution.

Example 1 Solve the simultaneous equations $x + 3y = 6$

$$x^2 + y^2 = 10$$

Solution Make one letter the subject of the linear equation: $x = 6 - 3y$

Substitute into the quadratic equation $(6 - 3y)^2 + y^2 = 10$

Solve … $10y^2 - 36y + 26 = 0$

$$2(y - 1)(5y - 13) = 0$$

… to get two solutions: $y = 1$ or 2.6

Substitute both back into the *linear* equation $x = 6 - 3y = 3$ or -1.8

Write answers in pairs: $(x, y) = (3, 1)$ or $(-1.8, 2.6)$

- You can't just square root the quadratic equation.
- You could have substituted for y instead of x (though in this case that would have taken longer – try to avoid fractions if you can).
- It is very easy to make mistakes here. Take great care over accuracy.
- It is remarkably difficult to *set* questions of this sort in such a way that *both* pairs of answers are nice numbers. Don't worry if, as in this example, only *one* pair of answers are nice numbers.

Questions like this appear in many GCSE papers. They are often, however, rather simple (sometimes the quadratic equations are restricted to those of the form $x^2 + y^2 = a$) and it is important to practice less convenient examples.

Exercise

Solve the following simultaneous equations.

1 $x^2 + xy = 12$
$3x + y = 10$

2 $x^2 - 4x + y^2 = 21$
$y = 3x - 21$

3 $x^2 + xy + y^2 = 1$
$x + 2y = -1$

4 $x^2 - 2xy + y^2 = 1$
$y = 2x$

5 $c^2 + d^2 = 5$
$3c + 4d = 2$

6 $x + 2y = 15$
$xy = 28$

7 $2x^2 + 3xy + y^2 = 6$
$3x + 4y = 1$

8 $2x^2 + 4xy + 6y^2 = 4$
$2x + 3y = 1$

9 $4x^2 + y^2 = 17$
$2x + y = 5$

10 $2x^2 - 3xy + y^2 = 0$
$x + y = 9$

11 $x^2 + 3xy + 5y^2 = 15$
$x - y = 1$

12 $xy + x^2 + y^2 = 7$
$x - 3y = 5$

13 $x^2 + 3xy + 5y^2 = 5$
$x - 2y = 1$

14 $4x^2 - 4xy - 3y^2 = 20$
$2x - 3y = 10$

15 $x^2 - y^2 = 11$
$x - y = 11$

16 $\frac{12}{x} + \frac{1}{y} = 3$
$x + y = 7$

2.12 Expanding More Than Two Binomials

You should already be able to expand algebraic expressions of the form $(ax + b)(cx + d)$.

e.g. $(2x + 1)(3x - 2) = 6x^2 - 4x + 3x - 2 = 6x^2 - x - 2$

e.g. $(5x + 4)(5x - 4) = 25x^2 - 20x + 20x - 16 = 25x^2 - 16$

We are now going to algebraic expressions of the form $(ax + b)(cx + d)(ex + f)$.

To simplify the product of three binomials, first expand any two of the brackets and then multiply this answer by each term in the third bracket

Example 1:

Expand and simplify $(x - 2)(2x + 3)(x + 7)$

$(x - 2)(2x + 3) = 2x^2 + 3x - 4x - 6$ ← First expand two of the brackets (You may prefer to use the grid method)

$= 2x^2 - x - 6$ ← Simplify

Now $(x - 2)(2x + 3)(x + 7) = (x + 7)(2x^2 - x - 6)$

$= x(2x^2 - x - 6) + 7(2x^2 - x - 6)$ ← Multiply your expansion by each term in the 3rd bracket

$= 2x^3 - x^2 - 6x + 14x^2 - 7x - 42$

$= 2x^3 + 13x^2 - 13x - 42$ ← Simplify

Example 2:

Show that $(2x + 5)(x - 1)(4x - 3) = 8x^3 + 6x^2 - 29x + 15$ for all values of x.

$(2x + 5)(x - 1) = 2x^2 - 2x + 5x - 5$ ← First expand any two of the brackets.

$= 2x^2 + 3x - 5$ ← Simplify

Now $(2x + 5)(x - 1)(4x - 3) = (4x - 3)(2x^2 + 3x - 5)$

$= 4x(2x^2 + 3x - 5) - 3(2x^2 + 3x - 5)$ ← Multiply your expansion by each term in the 3rd bracket

$= 8x^3 + 12x^2 - 20x - 6x^2 - 9x + 15$ ← Remember the minus outside the 2nd bracket changes each sign inside the 2nd bracket

$= 8x^3 + 6x^2 - 29x + 15$ ← Simplify

To simplify the product of four binomials, first expand any two of the brackets and then expand the other two brackets and then multiply these answers.

Example 3:

Expand and simplify $(x + 3)(x - 3)(2x + 1)(5x - 6)$

$(x + 3)(x - 3) = x^2 - 9$ ⟵ Expand two of the brackets

$(2x + 1)(5x - 6) = 10x^2 - 7x - 6$ ⟵ Expand the other two brackets

$(x + 3)(x - 3)(2x + 1)(5x - 6)$

$= (x^2 - 9)(10x^2 - 7x - 6)$ ⟵ Use the two expansions above

$= x^2(10x^2 - 7x - 6) - 9(10x^2 - 7x - 6)$ ⟵ Multiply the 2nd bracket by each term in the 1st bracket

$= 10x^4 - 7x^3 - 6x^2 - 90x^2 + 63x + 54$

$= 10x^4 - 7x^3 - 96x^2 + 63x + 54$ ⟵ Simplify

EXERCISE:

1. Expand and simplify

(a) $(x + 1)(x + 4)(x + 5)$

(b) $(2x + 7)(3x + 1)(x + 8)$

(c) $(x - 3)(x - 1)(2x - 3)$

(d) $(3x + 8)(x - 2)(2x - 5)$

(e) $(5x - 1)(2x + 5)(3x - 2)$

(f) $(4x + 1)(2x + 7)(4x - 1)$

(g) $(x + 4)^2 (3x - 7)$

(h) $(6x + 5)(2x - 1)^2$

(i) $(x - 1)(x + 1)(4x - 1)(2x - 5)$

(j) $(x + 5)^2(x - 2)^2$

2. Show that $(2x + 3)^3 = 8x^3 + 36x^2 + 54x + 27$ for all values of x.

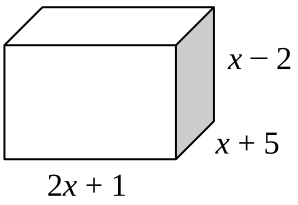
3. Show that $(x - 4)^2(x + 3)$ simplifies to $x^3 + ax^2 + bx + c$ where a, b and c are integers.

4. Express $(3x - 1)^4$ in the form $ax^4 + bx^3 + cx^2 + dx + e$ where a, b, c, d and e are integers.

5. $(3x + 5)(x - 4)(3x - 2) = 9x^3 + Ax^2 + Bx + 40$
Work out the value of A and the value of B.

6. $(x - 3)(2x + 1)(Ax + 1) = 8x^3 + Bx^2 + Cx - 3$
Work out the value of A, the value of B and the value of C.

7. Here is a cuboid.



All measurements are in centimetres.
Show that the volume of the cuboid is $(2x^3 + 7x^2 - 17x - 10)$ cm^3.

8. $f(x) = 3x^3 - 2x^2 + 4$

Express $f(x + 2)$ in the form $ax^3 + bx^2 + cx + d$.

9. The smallest of three consecutive positive odd numbers is $(2x - 1)$.

Work out the product of the three numbers.
Give your answer in the form $ax^3 + bx^2 + cx + d$.

2.13 Quadratic Inequalities

You should be able to solve quadratic equations of the form $ax^2 + bx + c = 0$

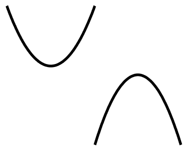
e.g. $x^2 - 3x - 4 = 0$ $(x - 4)(x + 1) = 0$ $x = 4$ or $x = -1$

e.g. $3x^2 - 14x + 8 = 0$ $(3x - 2)(x - 4) = 0$ $x = \frac{2}{3}$ or $x = 4$

e.g. $x^2 = 10 - 3x$ $x^2 + 3x - 10 = 0$ $(x + 5)(x - 2) = 0$ $x = -5$ or $x = 2$

You should also know the shape of a quadratic curve.

If the coefficient of x^2 is positive, the curve is '**smiling**'.
If the coefficient of x^2 is negative, the curve is '**frowning**'.



If $f(x) > 0$ or $f(x) \geq 0$ we want the values of x where $f(x)$ is **above** the x-axis.

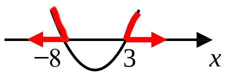
If $f(x) < 0$ or $f(x) \leq 0$ we want the values of x where $f(x)$ is **below** the x-axis.

Example 1:

Solve $x^2 + 5x - 24 \geq 0$

$(x + 8)(x - 3) \geq 0$ ← First factorise your quadratic expression

Critical values are $x = -8$ and $x = 3$ ← Solve $(x + 8)(x - 3) = 0$



← Always draw a sketch of your curve
Show where the curve cuts the x-axis
by solving $(x + 8)(x - 3) = 0$

$x \leq -8$ and $x \geq 3$ ← We want the area where $y \geq 0$

If you are asked to write the **solution set** of the inequality $x^2 + 5x - 24 \geq 0$ then the answer would be: $\{x : x \leq -8,\ x \geq 3\}$

NOTE: There are TWO regions so we write the answer as TWO inequalities.

Example 2:

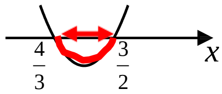
Find the solution set of the inequality $6(x^2 + 2) < 17x$

$6x^2 + 12 < 17x$ ← First expand the bracket

$6x^2 - 17x + 12 < 0$ ← Rearrange to the form $ax^2 + bx + c < 0$

$(3x - 4)(2x - 3) < 0$ ← Factorise in order to find where it cuts the x-axis

Critical values are $x = 4/3$ and $3/2$ ← Solve $(3x - 4)(2x - 3) = 0$



← Sketch the curve and shade below the axis

$\frac{4}{3} < x < \frac{3}{2}$ ← We want the region where f(x) is **below** the x-axis

Solution set = $\{ x : \frac{4}{3} < x < \frac{3}{2} \}$ ← Make sure your answer is given in the correct form

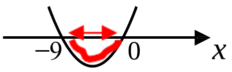
NOTE: There is ONE region so we write the answer as ONE inequality.

Example 3:

Solve $x(x + 9) \leq 0$

$x(x + 9) \leq 0$ ← This is already factorised with 0 on one side so there is no need to expand the brackets

Critical values are $x = 0$ and $x = -9$



← Sketch the curve and shade below the axis

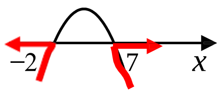
$-9 \leq x \leq 0$ ← We want the region where f(x) is **below** the x-axis. There is only one region so write as one inequality

Example 4:

Solve the inequality $14 + 5x < x^2$

$14 + 5x - x^2 < 0$ ← Rearrange to the form $ax^2 + bx + c < 0$

$(2 + x)(7 - x) < 0$ ← Factorise in order to find where it cuts the x-axis



← The curve is **'frowning'** as we have $-x^2$

$x < -2$ and $x > 7$ ← We want the region where f(x) is **below** the x-axis

OR

$14 + 5x - x^2 < 0$ ← Rearrange to the form $ax^2 + bx + c < 0$

$x^2 - 5x - 14 > 0$ ← Multiply each term by −1 which changes $<$ to $>$

$(x + 2)(x - 7) > 0$ ← Factorise in order to find where it cuts the x-axis

← The curve is **'smiling'** as we have $+x^2$

$x < -2$ and $x > 7$ ← This method gives the same answer as the 1st method

EXERCISE

1. Solve each of these inequalities.

(a) $x^2 + 9x + 18 \leq 0$ (b) $x^2 - x - 20 < 0$

(c) $(x - 2)(x + 7) > 0$ (d) $x^2 - 5x \geq 0$

(e) $2x^2 - 11x + 12 < 0$ (f) $(5 + x)(1 - 2x) \geq 0$

(g) $15 + 2x - x^2 \leq 0$ (h) $21 - x - 2x^2 > 0$

(i) $x(5x - 2) > 0$ (j) $x^2 - 2x > 35$

2. Find the solution set for each of these inequalities.

(a) $x^2 - 4x + 3 \leq 0$ (b) $x^2 + x - 42 < 0$

(c) $x(x + 2) > 48$ (d) $3x^2 + 14x - 5 \geq 0$

(e) $2x^2 > 11x - 12$ (f) $16 - x^2 \leq 6x$

(g) $7 + 2(4x^2 - 15x) \leq 0$ (h) $x^2 - 4(x + 6) > 8$

(i) $3x(5 - x) > 0$ (j) $(x + 5)^2 \geq 1$

3. Solve $\dfrac{x^2 + 12}{2} \geq 4x$

4. Find the solution set for which $15 + 2x \leq x^2$

5. Find the set of values for which $6 + x \geq x^2$ and $x + 2 < x^2$

6. Find the solution set for $(x - 3)(2x + 3) < 2x(1 - 2x) - 5$

2.14 Using Completing the Square to Find Turning Points

You should already be able to express a quadratic equation in the form $a(x + b)^2 + c$ by completing the square.

e.g. $x^2 - 6x + 3 = (x - 3)^2 - 9 + 3 = (x - 3)^2 - 6$

e.g. $3x^2 + 6x + 5 = 3[x^2 + 2x] + 5 = 3[(x + 1)^2 - 1] + 5 = 3(x + 1)^2 + 2$

We are now going to deduce the turning points of a quadratic function after completing the square.

Example 1:

Given $y = x^2 + 6x - 5$, by writing it in the form $y = (x + a)^2 + b$, where a and b are integers, write down the coordinates of the turning point of the curve. Hence sketch the curve.

$y = x^2 + 6x - 5$

$= (x + 3)^2 - 9 - 5$ ← Remember to halve the coefficient of x

$= (x + 3)^2 - 14$ and subtract $(-3)^2$ to compensate

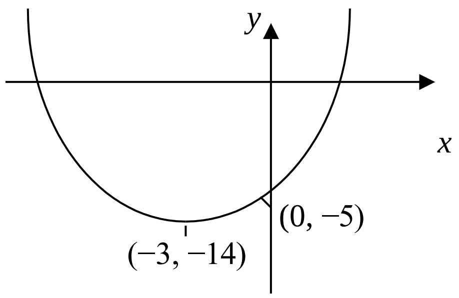
The turning point occurs when $(x + 3)^2 = 0$, i.e. when $x = -3$

When $x = -3$, $y = (-3 + 3)^2 - 14 = 0 - 14 = -14$

So the coordinates of the turning point is $(-3, -14)$

The graph $y = x^2 + 6x - 5$ cuts the y-axis when $x = 0$, i.e. $y = -5$

Sketch:



When $y = (x + a)^2 + b$ then the coordinates of the turning point is $(-a, b)$.

The minimum or maximum value of y is b.

Example 2:

Given that the minimum turning point of a quadratic curve is (1, −6), find an equation of the curve in the form $y = x^2 + ax + b$. Hence sketch the curve.

$y = (x - 1)^2 - 6$ ⟵ If the minimum is when $x = 1$, we know we have $(x - 1)^2$

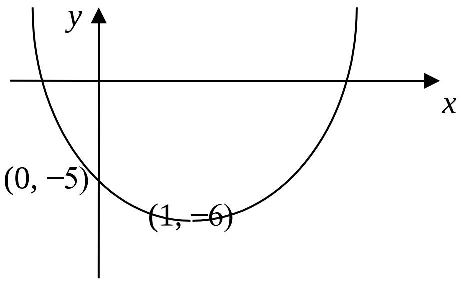
$= (x^2 - x - x + 1) - 6$ ⟵ If the minimum is when $y = -6$, we know we have $(...)^2 - 6$

$= x^2 - 2x - 5$

An equation of the curve is $y = x^2 - 2x - 5$

The graph cuts the y-axis when $x = 0$, i.e. at $y = -5$

Sketch: It is a minimum turning point so the shape is 



NOTE: There are other possible equations as, for example $y = 4(x - 1)^2 - 6$ also has a turning point of (1, −6). If it was a maximum turning point then the coefficient of x^2 would be negative.

Example 3:

Find the maximum value of $-x^2 + 4x - 7$ and sketch the curve.

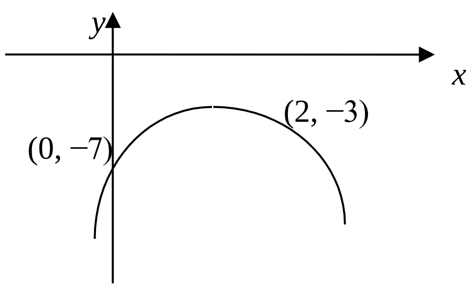
$-x^2 + 4x - 7 = -(x^2 - 4x + 7)$ ⟵ First take out the minus sign

$= -[(x - 2)^2 - 4 + 7]$ ⟵ Remember to use square brackets

$= -[(x - 2)^2 + 3]$

$= -(x - 2)^2 - 3$ ⟵ Multiply $(x - 2)^2$ and +3 by −1

The maximum value is −3



It is a maximum value so the shape is 

Exercise:

1. By writing the following in the form $y = (x + a)^2 + b$, where a and b are integers, write down the coordinates of the turning point of the curve. Hence sketch the curve.

(a) $y = x^2 - 8x + 20$
(b) $y = x^2 - 10x - 1$
(c) $y = x^2 + 4x - 6$
(d) $y = 2x^2 - 12x + 8$
(e) $y = -x^2 + 6x + 10$
(f) $y = 5 - 2x - x^2$

2. Given the following minimum turning points of quadratic curves, find an equation of the curve in the form $y = x^2 + ax + b$. Hence sketch each curve.

(a) $(2, -3)$
(b) $(-4, 1)$
(c) $(-1, 5)$
(d) $(3, -12)$
(e) $(1, -7)$
(f) $(-4, -1)$

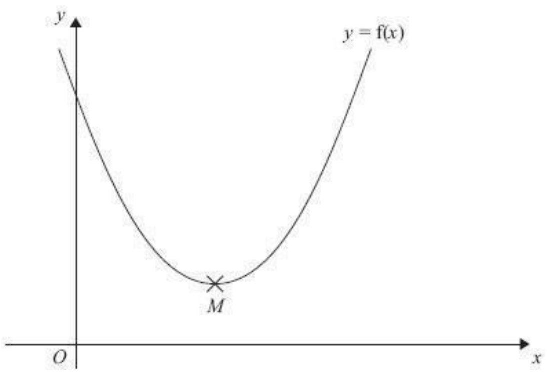
3. Find the maximum or minimum value of the following curves and sketch each curve.

(a) $y = x^2 + 4x + 2$
(b) $y = 1 - 6x - x^2$
(c) $y = -x^2 + 2x - 3$
(d) $y = x^2 - 8x + 8$
(e) $y = x^2 - 3x - 1$
(f) $y = -3x^2 + 12x - 9$

4. The expression $x^2 - 3x + 8$ can be written in the form $(x - a)^2 + b$ for all values of x.

(i) Find the value of a and the value of b.

The equation of a curve is $y = f(x)$ where $f(x) = x^2 - 3x + 8$
The diagram shows part of a sketch of the graph of $y = f(x)$.



The minimum point of the curve is M.

(ii) Write down the coordinates of M.

5. (i) Sketch the graph of $f(x) = x^2 – 6x + 10$, showing the coordinates of the turning point and the coordinates of any intercepts with the coordinate axes.
(ii) Hence, or otherwise, determine whether $f(x) − 3= 0$ has any real roots.
Give reasons for your answer.

***6.** The minimum point of a quadratic curve is $(1, −4)$. The curve cuts the y-axis at $−1$.
Show that the equation of the curve is $y = 3x^2 − 6x −1$

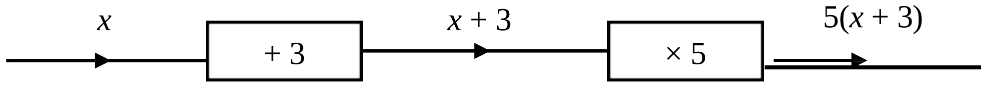
***7.** The maximum point of a quadratic curve is $(−2, −5)$. The curve cuts the y-axis at $−13$.
Find the equation of the curve. Give your answer in the form $ax^2 + bx + c$.

* = extension

2.15 Composite Functions

A **composite function** is a function consisting of 2 or more functions.

The term composition is used when one operation is performed after another operation.
For instance:



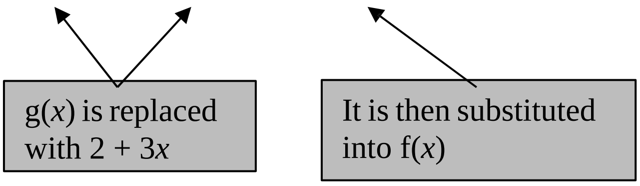
This function can be written as $f(x) = 5(x + 3)$

Suppose $f(x) = x^2$ and $g(x) = 2 + 3x$ **What is fg(*x*)?**

Now $fg(x) = f[g(x)]$

This means apply g first and then apply f.

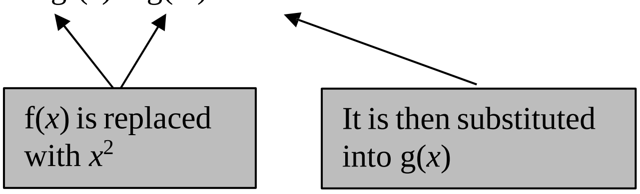
$fg(x) = f(2 + 3x) = (2 + 3x)^2$



What is gf(*x*)?

This means apply f first and then apply g.

$gf(x) = g(x^2) = 2 + 3x^2$



NOTE: The composite function **gf(*x*)** means apply f first followed by g.

NOTE: The composite function **fg(*x*)** means apply g first followed by f.

NOTE: $fg(x)$ can be written as fg and $gf(x)$ can be written as gf; fg is not the same as gf.

Example 1:

f and g are functions such that $f(x) = \frac{1}{x}$ and $g(x) = 3 - 2x$

Find the composite functions (a) fg (b) gf

(a) $fg = fg(x) = f(3 - 2x)$ ⟵ Do g first: Put $(3 - 2x)$ instead of $g(x)$

$= \frac{1}{3-2x}$ ⟵ Substitute $(3 - 2x)$ for x in $\frac{1}{x}$

(b) $gf = gf(x) = g\left(\frac{1}{x}\right)$ ⟵ Do f first: Put $\frac{1}{x}$ instead of $f(x)$

$= 3 - 2\frac{1}{x} = 3 - \frac{2}{x}$ ⟵ Substitute $\frac{1}{x}$ for x in $(3 - 2x)$

Example 2:

$f(x) = 7 - 2x$ $g(x) = 4x - 1$ $h(x) = 3(x - 1)$

Find the following composite functions: (a) gf (b) gg (c) fgh

(a) $gf = gf(x) = g(7 - 2x)$ ⟵ Do f first: Put $(7 - 2x)$ instead of $f(x)$

$= 4(7 - 2x) - 1$ ⟵ Substitute $(7 - 2x)$ for x in $4x - 1$

$= 27 - 8x$ ⟵ Simplify $28 - 8x - 1$

(b) $gg = gg(x) = g(4x - 1)$ ⟵ Put $(4x - 1)$ instead of $g(x)$

$= 4(4x - 1) - 1$ ⟵ Substitute $(4x - 1)$ for x in $4x - 1$

$= 16x - 5$ ⟵ Simplify $16x - 4 - 1$

(c) $fgh = fgh(x) = fg[3(x - 1)]$ ⟵ Put $3(x - 1)$ instead of $h(x)$

$= fg(3x - 3)$ ⟵ Expand $3(x - 1)$

$= f[4(3x - 3) - 1]$ ⟵ Substitute $(3x - 3)$ for x in $4x - 1$

$= f(12x - 13)$ ⟵ Simplify $12x - 12 - 1$

$= 7 - 2(12x - 13)$ ⟵ Substitute $(12x - 13)$ for x in $7 - 2x$

$= 33 - 24x$ ⟵ Simplify $7 - 24x + 26$

Example 3:

$f(x) = 7 - 2x$ $g(x) = 4x - 1$ $h(x) = 3(x - 1)$

Evaluate (a) fg (5) (b) ff (–2) (c) ghf (3)

(a) $fg(5) = f(20 - 1) = f(19)$ ⟵ Substitute for $x = 5$ in $4x - 1$

$= 7 - 2(19) = -31$ ⟵ Substitute for $x = 19$ in $7 - 2x$

(b) $ff(-2) = f[7 - 2(-2)] = f(11)$ ⟵ Substitute for $x = -2$ in $7 - 2x$ and simplify

$= 7 - 2(11) = -15$ ⟵ Substitute for $x = 11$ in $7 - 2x$ and simplify

(c) $ghf(3) = gh(7 - 6) = gh(1)$ ⟵ Substitute for $x = 3$ in $7 - 2x$ and simplify

$= g[3(1 - 1)] = g(0)$ ⟵ Substitute for $x = 1$ in $3(x - 1)$ and simplify

$= 4(0) - 1 = -1$ ⟵ Substitute for $x = 0$ in $4x - 1$ and simplify

Example 4:

$f(x) = 3x + 2$ and $g(x) = 7 - x$

Solve the equation $gf(x) = 2x$

$gf(x) = g(3x + 2)$ ⟵ Put $(3x + 2)$ instead of $f(x)$

$= 7 - (3x + 2)$ ⟵ g's rule is subtract from 7

$= 5 - 3x$ ⟵ Simplify $7 - 3x - 2$

$5 - 3x = 2x$ ⟵ Put $gf(x) = 2x$ and solve

$5 = 5x$ ⟵ Add $3x$ to both sides

$x = 1$

Example 5:

(more challenging question)

(a)

Functions f, g and h are such that

f: $x \longrightarrow 4x - 1$ g: $x \longrightarrow \frac{1}{x+2}, x \neq -2$ h: $x \longrightarrow (2-x)^2$

Find (a)(i) fg(x) (ii) hh(x) (b) Show that hgf(x) = $\left(\frac{8x+1}{4x+1}\right)^2$

(a) $\text{fg}(x) = \text{f}\left(\frac{1}{x+2}\right)$ ⟵ Substitute for g(x)

$= 4\left(\frac{1}{x+2}\right) - 1$ ⟵ f's operation is × 4 − 1

$= \frac{4 - 1(x+2)}{x+2}$ ⟵ Simplify using a common denominator of $x + 2$

$= \frac{2-x}{x+2}$

(b) $\text{hh}(x) = \text{h}(2-x)^2$ Substitute for h(x)

$= [2 - (2-x)^2]^2$ ⟵ h's operation is subtract from 2 and then square

$= [2 - (4 - 2x + x^2)]^2$ ⟵ $(2-x)^2 = (2-x)(2-x) = 4 - 2x + x^2$

$= (-2 + 4x - 2x^2)^2$

(c) $\text{hgf}(x) = \text{hg}(4x-1)$ ⟵ Put $(4x - 1)$ for f(x)

$= \text{h}\left(\frac{1}{4x-1+2}\right)$ ⟵ Substitute $(4x - 1)$ for x in g(x)

$= \left(2 - \frac{1}{4x+1}\right)^2$ ⟵ $4x - 1 + 2 = 4x + 1$, so put $4x + 1$ for x in h(x)

$= \left[2\left(\frac{4x+1}{4x+1}\right) - \frac{1}{4x+1}\right]^2$ ⟵ Simplify using a common denominator of $4x + 1$

$= \left(\frac{8x+2-1}{4x+1}\right)^2$

$= \left(\frac{8x+1}{4x+1}\right)^2$

Exercise:

1. Find an expression for fg(x) for each of these functions:

(a) $f(x) = x - 1$ and $g(x) = 5 - 2x$

(b) $f(x) = 2x + 1$ and $g(x) = 4x + 3$

(c) $f(x) = \frac{3}{x}$ and $g(x) = 2x - 1$

(d) $f(x) = 2x^2$ and $g(x) = x + 3$

2. Find an expression for gf(x) for each of these functions:

(a) $f(x) = x - 1$ and $g(x) = 5 - 2x$

(b) $f(x) = 2x + 1$ and $g(x) = 4x + 3$

(c) $f(x) = \frac{3}{x}$ and $g(x) = 2x - 1$

(d) $f(x) = 2x^2$ and $g(x) = x + 3$

3. The function f is such that $f(x) = 2x - 3$

Find (i) ff(2) (ii) Solve the equation $ff(a) = a$

4. Functions f and g are such that

$f(x) = x^2$ and $g(x) = 5 + x$

Find (a)(i) fg(x) (ii) gf(x)

(b) Show that there is a single value of x for which $fg(x) = gf(x)$ and find this value of x.

5. Given that $f(x) = 3x - 1$, $g(x) = x^2 + 4$ and $fg(x) = gf(x)$, show that $x^2 - x - 1 = 0$

6. The function f is defined by $f(x) = \frac{x - 1}{x}$, $x \neq 0$

Solve $ff(x) = -2$

7. The function g is such that $g(x) = \frac{1}{1 - x}$ for $x \neq 1$

(a) Prove that $gg(x) = \frac{x - 1}{x}$

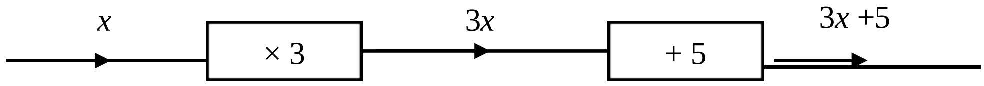
(b) Find ggg(3)

8. Functions f, g and h are such that $f(x) = 3 - x$, $g(x) = x^2 - 14$ and $h(x) = x - 2$

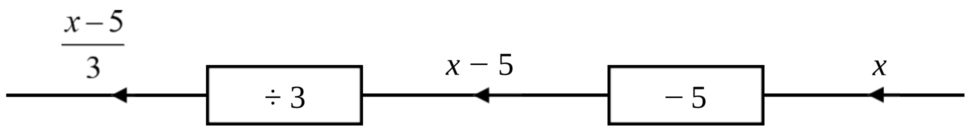
Given that $f(x) = gfh(x)$, find the values of x.

2.16 *Inverse Functions*

The function f(x) = $3x + 5$ can be thought of as a sequence of operations as shown below



Now reversing the operations



The new function, $\dfrac{x-5}{3}$, is known as the **inverse** function.

Inverse functions are denoted as $f^{-1}(x)$.

Example 1:

Find the inverse function of $f(x) = 3x - 4$

$y = 3x - 4$ ⟵ **Step 1:** Write out the function as $y = ...$

$x = 3y - 4$ ⟵ **Step 2:** Swap the x and y

$x + 4 = 3y$ ⟵ **Step 3:** Make y the subject

$\dfrac{x+4}{3} = y$

$f^{-1}(x) = \dfrac{x+4}{3}$ ⟵ **Step 4:** Instead of $y =$ write $f^{-1}(x) =$

Example 2:

Find the inverse function of $f(x) = \frac{x-2}{7}$

$y = \frac{x-2}{7}$ ⟵ **Step 1:** Write out the function as $y = ...$

$x = \frac{y-2}{7}$ ⟵ **Step 2:** Swap the x and y

$\left.\begin{array}{l} 7x = y - 2 \\ 7x + 2 = y \end{array}\right\}$ ⟵ **Step 3:** Make y the subject

$f^{-1}(x) = 7x + 2$ ⟵ **Step 4:** Instead of $y =$ write $f^{-1}(x) =$

Example 3:

Find the inverse function of $f(x) = \sqrt{x+4}$

$y = \sqrt{x+4}$ ⟵ **Step 1:** Write out the function as $y = ...$

$x = \sqrt{y+4}$ ⟵ **Step 2:** Swap the x and y

$\left.\begin{array}{l} x^2 = y + 4 \\ x^2 - 4 = y \end{array}\right\}$ ⟵ **Step 3:** Make y the subject

$f^{-1}(x) = x^2 - 4$ ⟵ **Step 4:** Instead of $y=$ write $f^{-1}(x) =$

RULES FOR FINDING THE INVERSE $f^{-1}(x)$:

Step 1: Write out the function as $y = ...$

Step 2: Swap the x and y

Step 3: Make y the subject

Step 4: Instead of $y=$ write $f^{-1}(x) =$

Exercise:

1. Find the inverse function, $f^{-1}(x)$, of the following functions:

(a) $f(x) = 3x - 1$

(b) $f(x) = 2x + 3$

(c) $f(x) = 1 - 2x$

(d) $f(x) = x^2 + 5$

(e) $f(x) = 6(4x - 1)$

(f) $f(x) = 4 - x$

(g) $f(x) = 3x^2 - 2$

(h) $f(x) = 2(1 - x)$

(i) $f(x) = \dfrac{2}{x+1}$

(j) $f(x) = \dfrac{x+1}{x-2}$

2. The function f is such that $f(x) = 7x - 3$

(a) Find $f^{-1}(x)$.

(b) Solve the equation $f^{-1}(x) = f(x)$.

3. The function f is such that $f(x) = \dfrac{8}{x+2}$

(a) Find $f^{-1}(x)$.

(b) Solve the equation $f^{-1}(x) = f(x)$.

4. The function f is such that $f(x) = \dfrac{1}{x+4}$, $x \neq -4$.

Evaluate $f^{-1}(3)$. [Hint: First find $f^{-1}(x)$ and then substitute for $x = -3$]

5. $f(x) = \dfrac{x}{x+3}$, $x \in \mathbb{R}$, $x \neq -3$

(a) If $f^{-1}(x) = -5$, find the value of x.

(b) Show that $ff^{-1}(x) = x$

6. Functions f and g are such that

$f(x) = 3x + 2$ $g(x) = x^2 + 1$

Find an expression for $(fg)^{-1}(x)$ [Hint: First find $fg(x)$]

2.17 Straight line graphs

I am sure that you are very familiar with the equation of a straight line in the form $y = mx + c$, and you have probably practised converting to and from the forms

$$ax + by + k = 0 \qquad \text{or} \qquad ax + by = k,$$

usually with a, b and k are integers. You need to be fluent in moving from one form to the other. The first step is usually to get rid of fractions by multiplying both sides by a common denominator.

Example 1 Write $y = \frac{3}{5}x - 2$ in the form $ax + by + k = 0$, where a, b and k are integers.

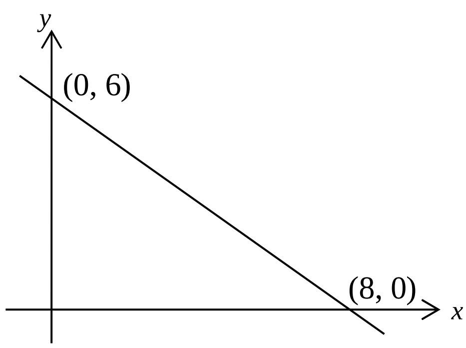
Solution Multiply both sides by 5: $5y = 3x - 10$
Subtract $5y$ from both sides: $0 = 3x - 5y - 10$
or $3x - 5y - 10 = 0$

In the first line it is a very common mistake to forget to multiply the 2 by 5.
It is a bit easier to get everything on the right instead of on the left of the equals sign, and this reduces the risk of making sign errors.

In plotting or sketching lines whose equations are written in the form $ax + by = k$, it is useful to use the *cover-up rule*:

Example 2 Draw the graph of $3x + 4y = 24$.

Solution Put your finger over the "$3x$". You see "$4y = 24$".
This means that the line hits the y-axis at (0, 6).
Repeat for the "$4y$". You see "$3x = 24$".
This means that the line hits the x-axis at (8, 0).
[NB: *not* the point (8, 6)!]
Mark these points in on the axes.
You can now draw the graph.



1 Rearrange the following in the form $ax + by + c = 0$ or $ax + by = c$ as convenient, where a, b and c are integers and $a > 0$.

(a) $y = 3x - 2$ (b) $y = \frac{1}{2}x + 3$

(c) $y = -\frac{3}{4}x + 3$ (d) $y = \frac{7}{2}x - \frac{5}{4}$

(e) $y = -\frac{2}{3}x + \frac{3}{4}$ (f) $y = \frac{4}{7}x - \frac{2}{3}$

2 Rearrange the following in the form $y = mx + c$. Hence find the gradient and the y-intercept of each line.

(a) $2x + y = 8$ (b) $4x - y + 9 = 0$

(c) $x + 5y = 10$ (d) $x - 3y = 15$

(e) $2x + 3y + 12 = 0$ (f) $5x - 2y = 20$

(g) $3x + 5y = 17$ (h) $7x - 4y + 18 = 0$

3 Sketch the following lines. Show on your sketches the coordinates of the intercepts of each line with the x-axis and with the y-axis.

(a) $2x + y = 8$

(b) $x + 5y = 10$

(c) $2x + 3y = 12$

(d) $3x + 5y = 30$

(e) $3x - 2y = 12$

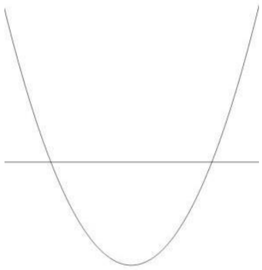
(f) $4x + 5y + 20 = 0$

2.18 *Factors*

Factors are crucial when curve-sketching.
They tell you where the curve meets the x-axis.
Do not multiply out brackets!

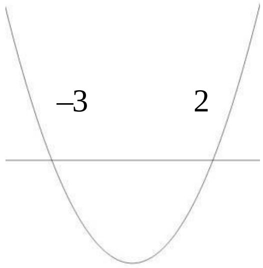
Example Sketch the graph of $y = (x - 2)(x + 3)$.

Solution The graph is a *positive* (happy!) *parabola* so start by drawing the *correct shape* with a *horizontal axis* across it.



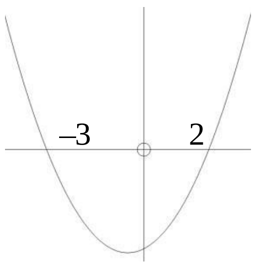
The factors tell you that it hits the x-axis at $x = -3$ and $x = 2$.

Mark these on your sketch:



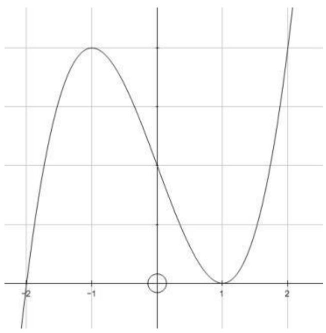
and only now put in the y-axis, which is clearly slightly nearer 2 than –3:

Note: the lowest point on the graph is *not* on the y-axis. (Because the graph is symmetric, it is at $x = -\frac{1}{2}$.)



Repeated factors

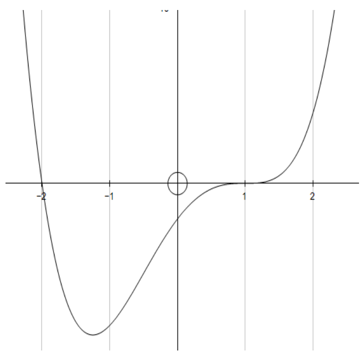
Suppose you want to sketch $y = (x - 1)^2(x + 2)$.
You know there is an intercept at $x = -2$.
At $x = 1$ the graph *touches* the axes, as if it were the graph of $y = (x - 1)^2$ there.



[More precisely, it is very like $y = 3(x - 1)^2$ there. That is because, close to $x = 1$, the $(x - 1)^2$ factor changes rapidly, while $(x + 2)$ remains close to 3.]

Likewise, the graph of $y = (x + 2)(x - 1)^3$ looks like $y = (x - 1)^3$ close to $x = 1$.

[Again, more precisely, it is very like $y = 3(x - 1)^3$ there.]



Sketch the curves in questions 1–21. Use a different diagram for each.
Show the x-coordinates of the intersections with the x-axis.

1 $y = x^2$

2 $y = (x - 1)(x - 3)$

3 $y = (x + 2)(x - 4)$

4 $y = x(x - 3)$

5 $y = (x + 2)(3x - 2)$

6 $y = x(4x + 3)$

7 $y = -x(x - 3)$

8 $y = (2 - x)(x + 1)$

9 $y = (3 - x)(2 + x)$

10 $y = (x + 2)(x - 1)(x - 4)$

11 $y = x(x - 1)(x + 2)$

12 $y = -x(x - 1)(x + 2)$

13 $y = (3 - x)(2 - x)(1 - x)$

14 $y = (x - 1)^2(x - 3)$

15 $y = (x - 1)(x - 3)^2$

16 $y = (x + 1)^3$

17 $y = (2 - x)(x + 1)^3$

18 $y = (x + 1)(x + 2)(x - 1)(x - 2)$

19 $y = -(x + 3)(x + 2)(x - 1)(x - 4)$

20 $y = (x - 2)^2(x + 2)^2$

21 $y = (x - 1)(x - 2)^2(x - 3)^3$

22 (a) Sketch the graph of $y = x^2$.

(b) Sketch $y = 2x^2$ on the same axes.

(c) Sketch $y = x^2 + 1$ on the same axes.

23 (a) Sketch the graph of $y = \sqrt{x}$.

(b) Sketch $y = 2\sqrt{x}$ on the same axes.

24 (a) Sketch the graph of $y = \frac{1}{x}$.

$\frac{\;}{x}$

25 (a) Sketch the graph of $y = \frac{1}{x^2}$.

(b) Sketch $y = \frac{2}{x^2}$ on the same axes.

26 (a) Sketch the graph of $y = x^3$.

(b) Sketch $y = 2x^3$ on the same axes.

27 (a) Sketch the graph of $y = x^4$.

(b) Sketch $y = 3x^4$ on the same axes.

28 (a) Sketch the graph of $y = x^3 - 4x$.

[Hint: It cuts the x-axis at –2, 0 and 2.]

(b) Sketch $y = 2x^3 - 8x$ on the same axes.

29 (a) Sketch the graph of $y = x^4 - x^2$.

[Hint: It cuts the x-axis at 1 and –1, and touches the axis at 0.]

(b) Sketch $y = -x^4 + x^2$ on the same axes.

30 Sketch, on separate axes, the following graphs. Show the x-coordinates of the intersections with the x-axis.

(a) $y = 4 - x^2$

(b) $y = (x - 2)(x + 1)$

(c) $y = -(x - 2)(x + 1)$

(d) $y = x(x + 4)$

(e) $y = (x - 2)^2$

(f) $y = -(x + 1)^2$

(g) $y = (1 - x)(2 + x)$

2.19 Trigonometry

The following two aspects are worth emphasising at this stage.

Trigonometric Equations

You can of course get one solution to an equation such as $\sin x = -0.5$ from your calculator. But what about others?

Example 1 Solve the equation $\sin x° = -0.5$ for $0 \le x < 360$.

Solution The calculator gives $\sin^{-1}(0.5) = -30$.

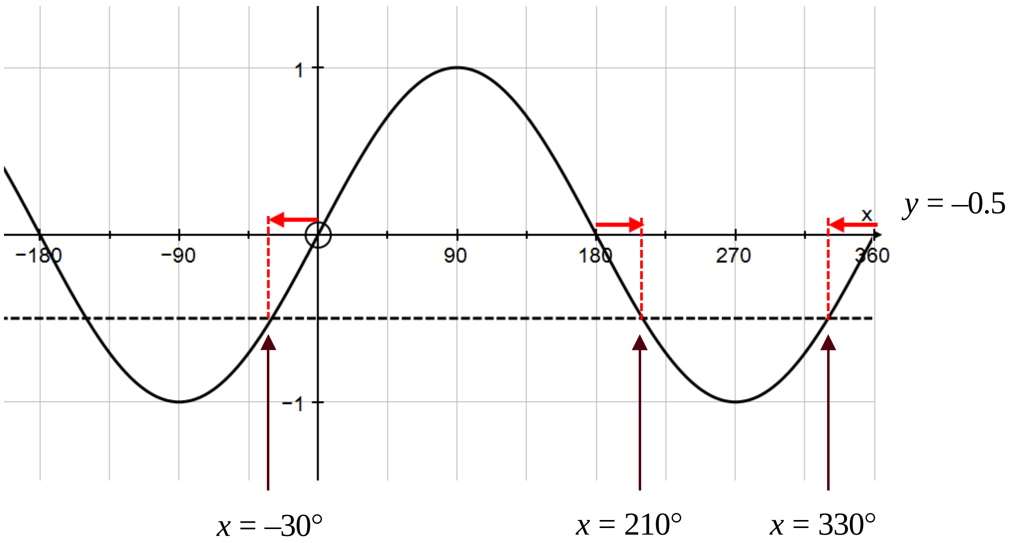
This is usually called the *principal value* of the function $\sin^{-1}$.

To get a second solution you can either use a graph or a standard rule.

Method 1: Use the graph of $y = \sin x$

By drawing the line $y = -0.5$ on the same set of axes as the graph of the sine curve, points of intersection can be identified in the range

$$0 \le x < 360.$$



Method 2: Use an algebraic rule.

To find the second solution you use $\sin (180 - x)° = \sin x°$

$$\tan (180 + x)° = \tan x°$$

$$\cos (360 - x)° = \cos x°.$$

Any further solutions are obtained by adding or subtracting 360 from the principal value or the second solution.

In this example the principal solution is –30°.

Therefore, as this equation involves sine, the second solution is:

$$180 - (-30)° = 210°$$

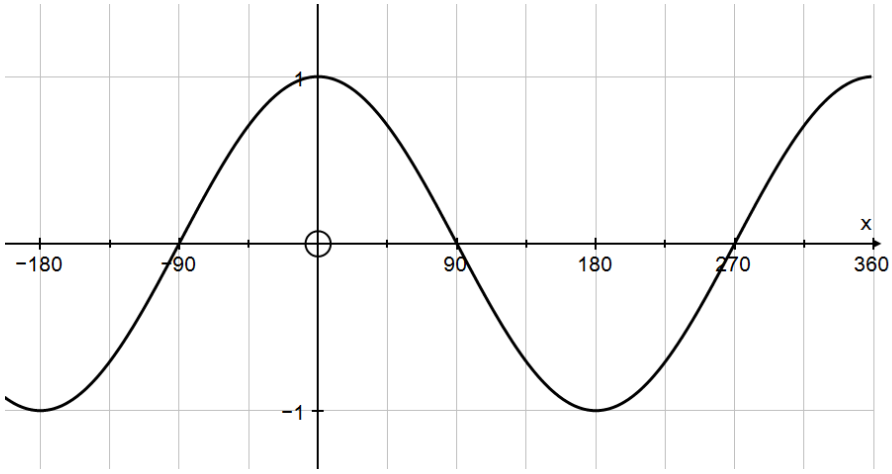
–30° is not in the required range, so add 360 to get:

$$360 + (-30) = 330°.$$

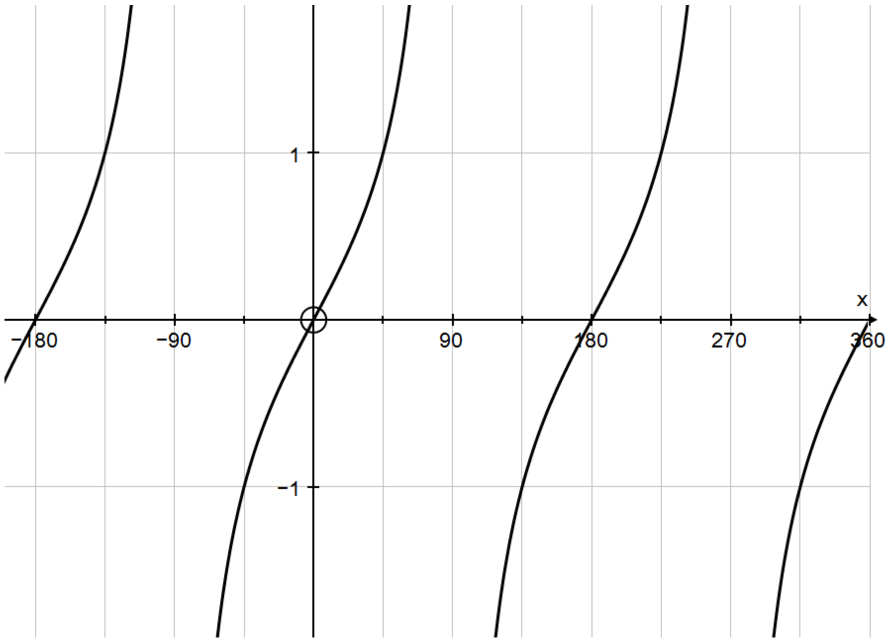
Hence the required solutions are 210° or 330°.

You should decide which method you prefer. The corresponding graphs for cos x and tan x are shown below.

$y = \cos x$



$y = \tan x$



To solve equations of the form $y = \sin(kx)$, you will expect to get $2k$ solutions in any interval of 360°. You can think of compressing the graphs, or of using a wider initial range.

Exercise

1 Solve the following equations for $0 \le x < 360$. Give your answers to the nearest 0.1°.

(a) $\sin x° = 0.9$ (b) $\cos x° = 0.6$ (c) $\tan x° = 2$

(d) $\sin x° = –0.4$ (e) $\cos x° = –0.5$ (f) $\tan x° = –3$

2 Solve the following equations for $–180 \le x < 180$. Give your answers to the nearest 0.1°.

(a) $\sin x° = 0.9$ (b) $\cos x° = 0.6$ (c) $\tan x° = 2$

(d) $\sin x° = –0.4$ (e) $\cos x° = –0.5$ (f) $\tan x° = –3$

Section 3. **Practice Booklet Test**

Complete the following, mark your work and send an image of this section, including your total mark, to Mr Steinhobel. You may NOT use a calculator

1. Expand and simplify
   (a) $(2x + 3)(2x - 1)$ (b) $(a + 3)^2$ (c) $4x(3x - 2) - x(2x + 5)$

2. Factorise
   (a) $x^2 - 7x$ (b) $y^2 - 64$ (c) $2x^2 + 5x - 3$ (d) $6t^2 - 13t + 5$

3. Simplify
   (a) $\frac{4x^3y}{8x^2y^3}$ (b) $\frac{3x+2}{3} + \frac{4x-1}{6}$ (c) $\frac{2}{x-2} + \frac{4}{x+3}$

4. Solve the following equations
   (a) $\frac{h-1}{4} + \frac{3h}{5} = 4$ (b) $x^2 - 8x = 0$ (c) $p^2 + 4p = 12$

5. If $2^{2x+1} \times 4^{x+1} = 8^{x+2}$, find the value of x.

6. a) Solve the simultaneous equations $3x - 5y = -11$
   $5x - 2y = 7$

   b) Solve the simultaneous equations $x^2 + y^2 = 25$
   $x + y = -1$

7. Rearrange the following equations to make x the subject
   (a) $v^2 = u^2 + 2ax$ (b) $V = \frac{1}{3}\pi x^2 h$ (c) $y = \frac{x+2}{x+1}$

8. Solve $5x^2 - x - 1 = 0$ giving your solutions in surd form

9. Find the values of x which satisfy the following inequalities
   (a) $5x - 2 < 6$ (b) $4 - 2x > 9$ (c) $x^2 - 6x - 16 > 0$

10. Given $(x) = 7x - 2$ and $g(x) = 2x^2 + 5x - 1$, find
    (a) (3) (b) $g(3)$ (c) $g(-4)$ (d) $fg(-2)$

11. Simplify
    (a) $\sqrt{18} \times \sqrt{75}$ (b) $\sqrt{54} + \sqrt{27}$ (c) $\frac{5}{\sqrt{2}}$ (d) $\frac{3-\sqrt{2}}{\sqrt{6}}$ (e) $\frac{1-\sqrt{2}}{3+\sqrt{2}}$

12. Solve for $-360° \leq \leq 360°$. (a) $\sin\theta = 0.2$ (b) $\cos\theta = -0.35$ (c) $\tan\theta = 12.6$

Solutions to the Exercises

2.1 Equations Containing Fractions

1) 7
2) 15
3) 24/7
4) 35/3
5) 3
6) 2
7) 9/5
8) 5

2.2 Forming Equations

1) 2n + 2n + 2 + 2n + 4 = 108 solve that to find n = 17. Sub into 2n, 2n +2 and 2n + 4 to find the 3 numbers.
2) x = 9.875 cm and 3x = 29.625 cm
3) Simultaneous equations eq 1. a + b = 72, eq 2. a-11 = 0.5 (b+11), solve to get 24 and 48

2.3 Simultaneous Equations

1) $x = 1$, $y = 3$
2) $x = -3$, $y = 1$
3) $x = 0$, $y = -2$
4) $x = 3$, $y = 1$
5) $a = 7$, $b = -2$
6) $p = 11/3$, $q = 4/3$

2.4 Factorising Quadratic Expressions

1) $(x-3)(x+2)$
2) $(x+8)(x-2)$
3) $(2x+1)(x+2)$
4) $x(2x-3)$
5) $(3x-1)(x+2)$
6) $(2y+3)(y+7)$
7) $(7y-3)(y-1)$
8) $5(2x-3)(x+2)$
9) $(2x+5)(2x-5)$
10) $(x-3)(x-y)$
11) $4(x-2)(x-1)$
12) $(4m-9n)(4m+9n)$
13) $y(2y-3(a)(2y+3(a)$
14) $2(4x-1)(x+2)$

2.5 Completing the Square

1 (a) $(x+4)^2+3$ (b) $(x-5)^2-2$ (c) $(x+1)^2-5$

(d) $(x-2)^2-7$ (e) $(x-1\frac{1}{2})^2-\frac{1}{4}$ (f) $(x-2\frac{1}{2})^2-12\frac{1}{4}$

2 (a) $3(x+1)^2+4$ (b) $5(x-2)^2-3$ (c) $2(x+2\frac{1}{2})^2+\frac{1}{2}$

3 (a) $(2x+3)^2+5$ (b) $(3x-2)^2-5$ (c) $(4x+5)^2-3$

2.6 Solving Quadratic Equations

1) (a) -1, -2 (b) -1, 4 (c) -5, 3

2) (a) 0, -3 (b) 0, 4 (c) 2, -2

3) (a) -1/2, 4/3 (b) 0.5, 2.5

4) (a) -5.30, -1.70 (b) 1.07, -0.699 (c) -1.20, 1.45

(d) no solutions (e) no solutions f) no solutions

2.7 Changing the Subject of a Formula

Ex B

1) $t=\frac{32rP}{w}$

2) $t=\pm\sqrt{\frac{32rP}{w}}$

3) $t=\pm\sqrt{\frac{3V}{\pi h}}$

4) $t=\frac{P^2g}{2}$

5) $t=v-\frac{Pag}{w}$

6) $t=\pm\sqrt{\frac{r-a}{b}}$

Ex C

1) $x=\frac{c-3}{a-b}$

2) $x=\frac{3a+2k}{k-3}$

3) $x=\frac{2y+3}{5y-2}$

4) $x=\frac{ab}{b-a}$

2.8 Indices

Ex A

1) $5b^6$
2) $6c^7$
3) b^3c^4
4) $-12n^8$
5) $4n^5$
6) d^2
7) a^6
8) $-d^{12}$

Ex B

1) 2
2) 3
3) 1/3
4) 1/25
5) 1
6) 1/7
7) 9
8) 9/4
9) ¼
10) 0.2
11) 4/9
12) 64
13) $6a^3$
14) x
15) xy^2

2.9 Surds

Ex A

1) $5\sqrt{2}$
2) $6\sqrt{2}$
3) $3\sqrt{3}$
4) $4\sqrt{5}$
5) $6\sqrt{10}$
6) $10\sqrt{3}$

Ex B

1) $\sqrt{21}$
2) $20\sqrt{10}$
3) $18\sqrt{2}$
4) $6\sqrt{6}$
5) $\frac{5}{3}$
6) 6
7) $7 + 6\sqrt{2}$
8) $5\sqrt{2} - 40 - \sqrt{6} + 8\sqrt{3}$

Ex C

1) $\sqrt{3} + \sqrt{7}$
2) $9\sqrt{2}$
3) $5\sqrt{6}$
4) $7\sqrt{2}$
5) $8\sqrt{3}$
6) $\sqrt{5}$
7) $\sqrt{2}$
8) $7\sqrt{3}$
9) $3\sqrt{2} + 3\sqrt{10}$
10) $6\sqrt{2} + \sqrt{3}$

Ex D

1

a) $\frac{\sqrt{2}}{2}$
b) $\frac{3\sqrt{5}}{5}$
c) $2\sqrt{5}$
d) $\frac{5\sqrt{7}}{14}$
e) $\frac{\sqrt{6}}{2}$
f) $\sqrt{10}$
g) $\frac{4\sqrt{3}+\sqrt{21}}{3}$
h) $3\sqrt{2} + 4\sqrt{10}$
i) $\frac{6\sqrt{5}-5}{5}$

2

a) $\sqrt{2} + 1$
b) $\sqrt{6} + 2$
c) $2(\sqrt{7} - 2)$
d) $\frac{1}{4}(3 - \sqrt{5})$
e) $\sqrt{6} + \sqrt{5}$

2.10 Algebraic Fractions

1 (a) $\frac{3x+5}{(x-1)(x+3)}$ (b) $\frac{x+7}{(x-3)(x+2)}$ (c) $\frac{x+3}{(2x-1)(3x+2)}$

(d) $\frac{x+5}{x+2}$ (e) $\frac{2x-3}{x-1}$ (f) $-\frac{x+3}{x+1}$

(g) $\frac{6x-1}{4(2x-1)(2x+1)}$

2 (a) $\frac{2x+1}{\sqrt{x}}$ (b) $\frac{3x+3}{\sqrt{x+3}}$ (c) $\frac{2x-2}{\sqrt[3]{x-2}}$

2.11 Linear and Quadratic Simultaneous Equations

1 (2, 4), (3, 1)

2 (6, –3), (7, 0)

3 (1, –1), (–1, 0)

4 (1, 2), (–1, –2)

5 (2, –1), $\left(-\frac{38}{25}, \frac{41}{25}\right)$

6 (7, 4), $\left(8, \frac{7}{2}\right)$

7 (–5, 4), $\left(\frac{19}{5}, -\frac{13}{5}\right)$

8 (–1, 1), $\left(\frac{5}{3}, -\frac{7}{9}\right)$

9 (2, 1), $\left(\frac{1}{2}, 4\right)$

10 (3, 6), $\left(-\frac{9}{2}, -\frac{9}{2}\right)$

11 (2, 1), $\left(-\frac{5}{9}, -\frac{14}{9}\right)$

12 (–1, –2), $\left(\frac{38}{13}, -\frac{9}{13}\right)$

13 $\left(\frac{5}{3}, \frac{1}{3}\right)$, $\left(-\frac{3}{5}, -\frac{4}{5}\right)$

14 (2, –2) (only)

15 (6, –5) (only)

16 (6, 1), $\left(\frac{14}{3}, \frac{7}{3}\right)$

2.12 Expanding More Than Two Binomials

1. (a) $x^3 + 10x^2 + 29x + 20$ (b) $6x^3 + 71x^2 + 191x + 56$
   (c) $2x^3 - 11x^2 + 18x - 9$ (d) $6x^3 - 11x^2 - 42x + 80$
   (e) $30x^3 + 49x^2 - 61x + 10$ (f) $32x^3 + 112x^2 - 2x - 7$
   (g) $3x^3 + 17x^2 - 8x - 112$ (h) $24x^3 - 4x^2 - 14x + 5$
   (i) $8x^4 - 22x^3 - 3x^2 + 22x - 5$ (j) $x^4 + 6x^3 - 11x^2 - 60x + 100$
2. Proof
3. $x^3 - 5x^2 - 8x + 48$
4. $81x^4 - 108x^3 + 54x^2 - 12x + 1$
5. $A = -27$ $B = -46$
6. $A = 4$ $B = -18$ $C = -17$.
7. Proof
8. $3x^3 + 16x^2 + 28x + 20$
9. $8x^3 + 12x^2 - 2x - 3$

2.13 Quadratic Inequalities

1. (a) $-6 \le x \le -3$ (b) $-4 < x < 5$
   (c) $x < -7$ or $x > 2$ (d) $x \le 0$ or $x \ge 5$
   (e) $\frac{3}{2} < x < 4$ (f) $-5 \le x \le \frac{1}{2}$
   (g) $x \le -3$ or $x \ge 5$ (h) $-\frac{7}{2} < x < 3$
   (i) $x < 0$ or $x > \frac{2}{5}$ (j) $x < -5$ or $x > 7$

2. (a) $\{ x : 1 \le x \le 3 \}$ (b) $\{ x : -7 < x < 6 \}$

(c) $\{ x : x < -8,\ x > 6 \}$ (d) $\{ x : x \le -5,\ x \ge \frac{1}{3} \}$

(e) $\{ x : x < \frac{3}{2},\ x > 4 \}$ (f) $\{ x : x \le -8, x \ge 2 \}$

(g) $\{ x : \frac{1}{4} \le x \le \frac{7}{2} \}$ (h) $\{ x : x < -4,\ x > 8 \}$

(i) $\{ x : 0 < x < 5 \}$ (j) $\{ x : x \le -6,\ x \ge -4 \}$

3. $x \le 2$ or $x \ge 6$

4. $\{ x : x \le -3,\ x \ge 5 \}$

5. $-2 \le x < -1$ and $2 < x \le 3$

6. $\{ x : -\frac{1}{2} < x < \frac{4}{3} \}$

2.14 Using Completing the Square to Find Turning Points

1. (a) (4, 4) (b) (5, −26)

(c) (−2, −10) (d) (3, −10)

(e) (3, 19) (f) (−1, 6)

2. (a) $y = x^2 - 4x + 1$ (b) $y = x^2 + 8x + 17$

(c) $y = x^2 + 2x + 6$ (d) $y = x^2 - 6x - 3$

(e) $y = x^2 - 2x - 6$ (f) $y = x^2 - 8x + 15$

3. (a) minimum (−2, −2) (b) maximum (−3, 10)

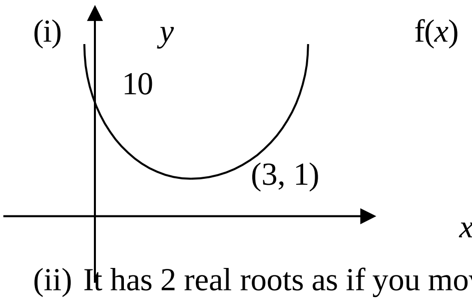
(c) maximum (1, −2) (d) minimum (4, −8)

(e) minimum (1.5, −3.25) (f) maximum (−2, 3)

4. (i) $a = 1.5$ $b = 5.75$

(ii) (1.5, 5.75)

5. (i)



(ii) It has 2 real roots as if you move the graph 3 down it will cut the x-axis twice as the minimum point will be (3, −2)

6. Minimum point is (1, −4) thus $y = A(x-1)^2 - 4 = Ax^2 - 2Ax + A - 4$

Cuts y-axis at −1, thus $A - 4 = -1$

$A = 3$ $y = 3x^2 - 2(3)x + 3 - 4$

$y = 3x^2 - 6x - 1$

7. $y = -2x^2 - 8x - 13$

2.15 Composite Functions

1. (a) $4 - 2x$ (b) $8x + 7$ (c) $\frac{3}{2x-1}$ (c) $2(x+3)^2$
2. (a) $7 - 2x$ (b) $8x + 7$ (c) $\frac{6}{x} - 1$ (d) $2x^2 + 3$
3. (a) -1 (b) $a = 3$

4. (a)(i) $(5+x)^2$ (ii) $5 + x^2$ (b) $x = -2$
5. $3(x^2 + 4) - 1 = (3x - 1)^2 + 4$
   $3x^2 + 12 - 1 = 9x^2 - 6x + 1 + 4$
   $3x^2 + 11 = 9x^2 - 6x + 5$
   $6x^2 - 6x - 6 = 0$
   $x^2 - x - 1 = 0$
6. $x = \frac{3}{2}$
7. (a) $g\left(\frac{1}{1-x}\right) = \frac{1}{1-\frac{1}{1-x}} = \frac{1}{\frac{1-x-1}{1-x}} = \frac{1-x}{-x} = \frac{x-1}{x}$ (b) 3
8. $x = 1$ and $x = 8$

2.16 Inverse Functions

1. (a) $\frac{x+1}{3}$ (b) $\frac{x-3}{2}$ (c) $\frac{1-x}{2}$ (d) $\pm\sqrt{x-5}$
   (e) $\frac{x+6}{24}$ (f) $4 - x$ (g) $\pm\sqrt{\frac{x+2}{3}}$ (h) $\frac{2-x}{2}$
   (i) $\frac{2-x}{x}$ (j) $\frac{2x+1}{x-1}$
2. (a) $\frac{x+3}{7}$ (b) $x = 0.5$
3. (a) $\frac{8}{x} - 2$ (b) $x = -4$ and $x = 2$
4. $-\frac{13}{3}$
5. (a) $\frac{5}{2}$ (b) $f\left(\frac{3x}{1-x}\right) = \frac{\frac{3x}{1-x}}{\frac{3x}{1-x}+3} = \frac{\frac{3x}{1-x}}{\frac{3x+3(1-x)}{1-x}} = \frac{\frac{3x}{1-x}}{\frac{3x+3-3x}{1-x}} = \frac{3x}{3} = x$
6. $\pm\sqrt{\frac{x-5}{3}}$

2.17 Straight Line Graphs

1 (a) $3x - y = 2$ (b) $x - 2y + 6 = 0$

(c) $3x + 4y = 12$ (d) $14x - 4y = 5$

(e) $8x + 12y = 9$ (f) $12x - 21y = 14$

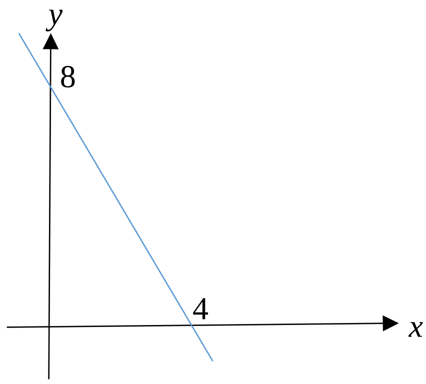
2 (a) $y = -2x + 8$; -2, 8 (b) $y = 4x + 9$; 4, 9

(c) $y = -\frac{1}{}x + 2$; $-\frac{1}{}$, 2 (d) $y = \frac{1}{}x - 5$; $\frac{1}{}$, -5

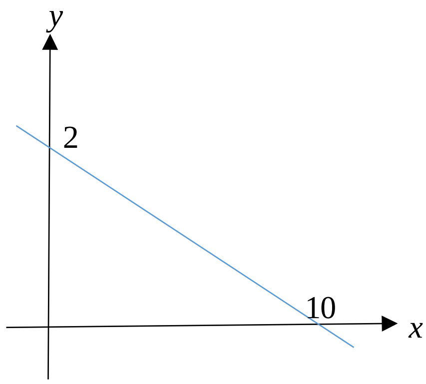
(e) $y = -\frac{2}{}x - 4$; $-\frac{2}{}$, -4 (f) $y = \frac{5}{}x - 10$; $\frac{5}{}$, -10

(b) $y = -\frac{3}{5}x + \frac{17}{5}$; $-\frac{3}{5}$, $\frac{17}{5}$ (h) $y = \frac{7}{4}x + \frac{9}{2}$; $\frac{7}{4}$, $\frac{9}{2}$

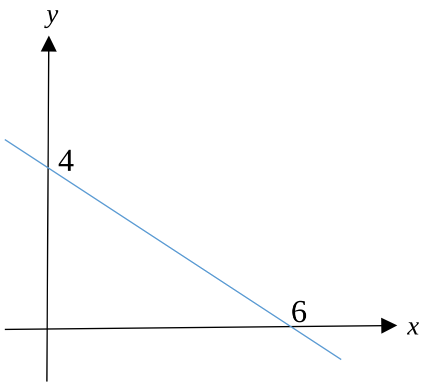
3 (a)



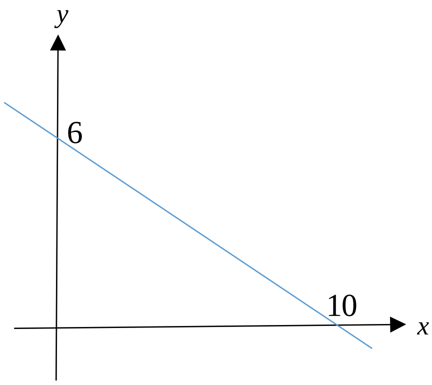
(b)



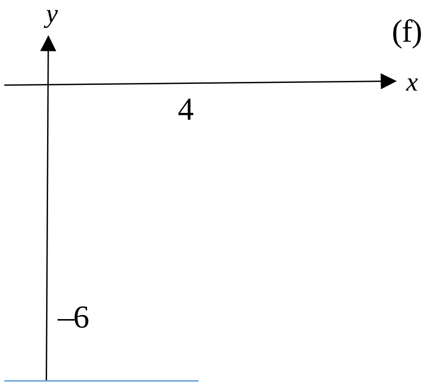
(c)



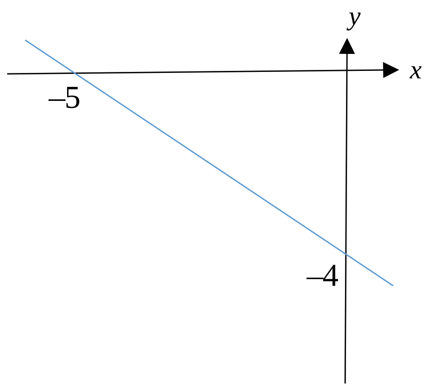
(d)



(e)

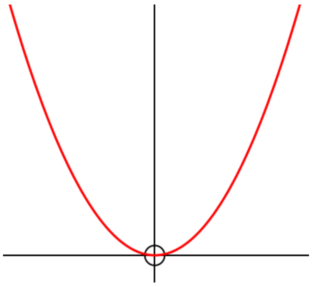


(f)



2.18 Factors

1



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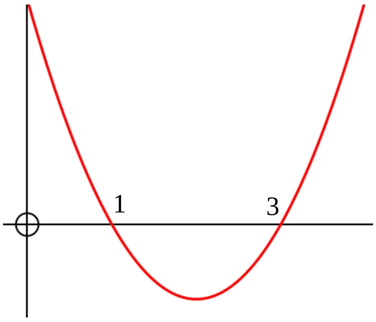

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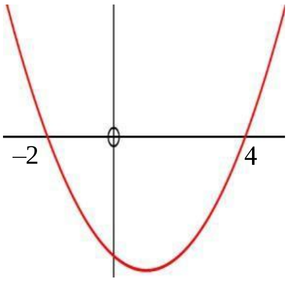

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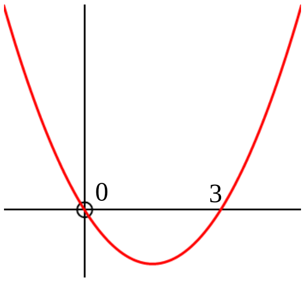

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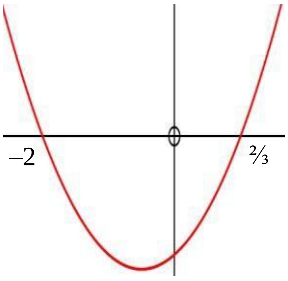

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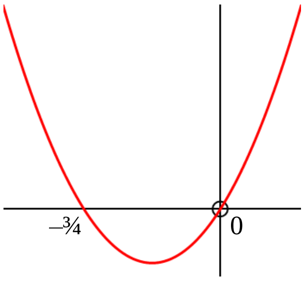

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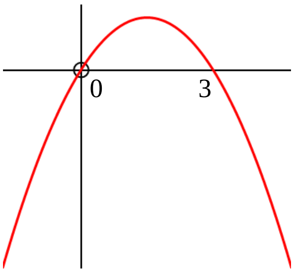

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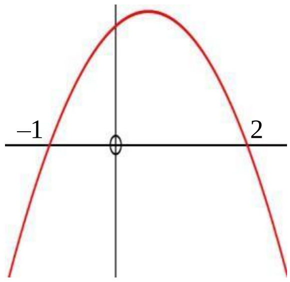

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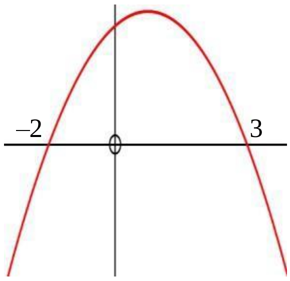

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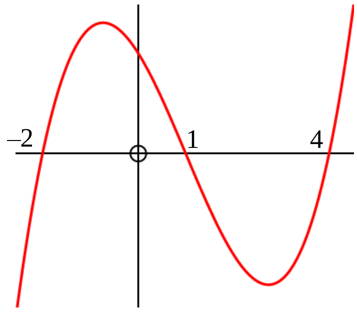

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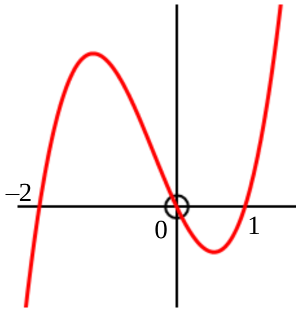

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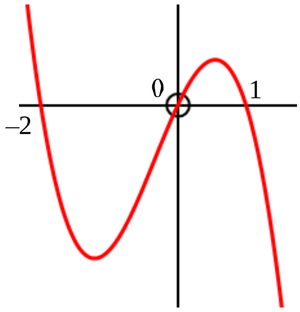

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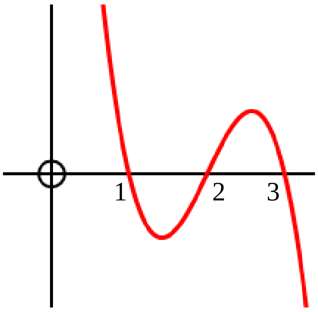

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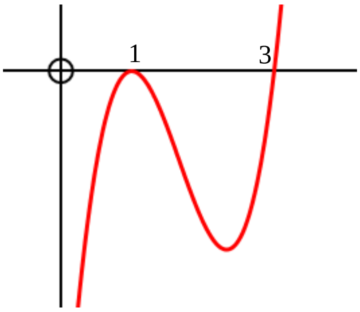

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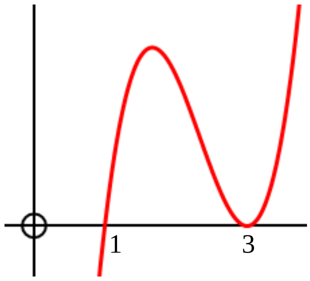

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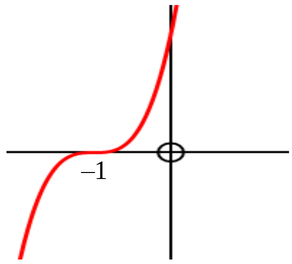

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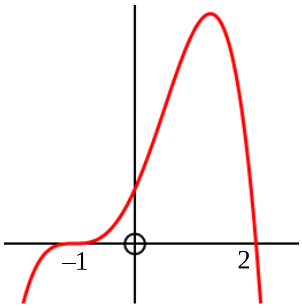

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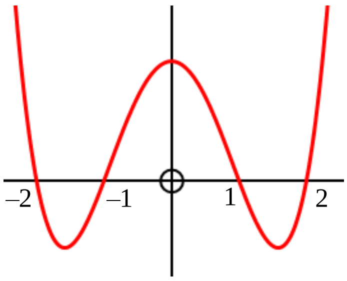

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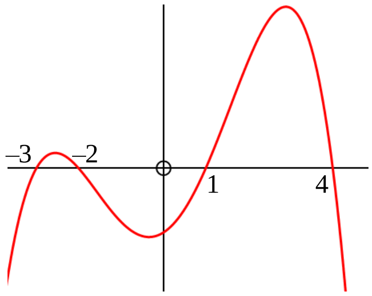

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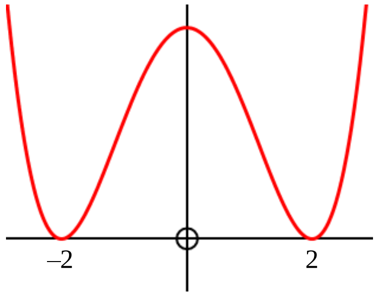

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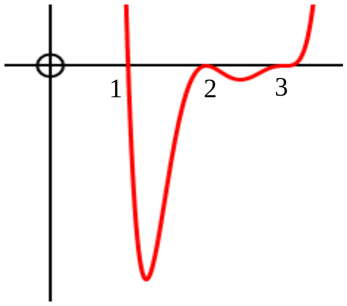
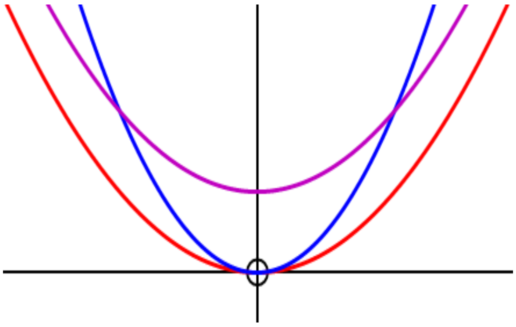


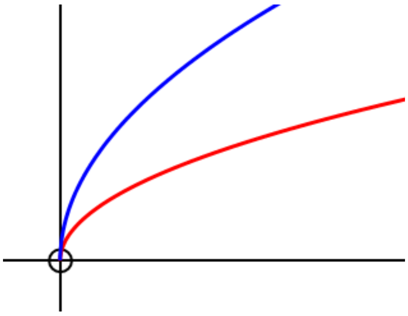
[In this graph in particular, do NOT worry about the y-coordinates of the minimum points.]

22



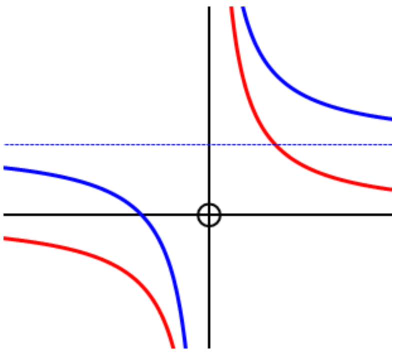
red: $y = x^2$
blue: $y = 2x^2$
purple: $y = x^2 + 1$

23



red: $y = \sqrt{x}$
blue: $y = 2\sqrt{x}$

24

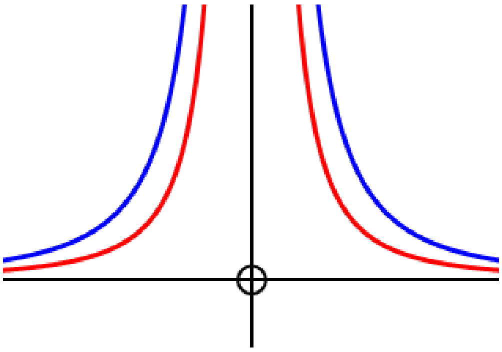


red: $y = \frac{1}{x}$

blue: $y = \frac{1}{x} + 1$

blue dotted: $y = 1$ *[horizontal asymptote]*

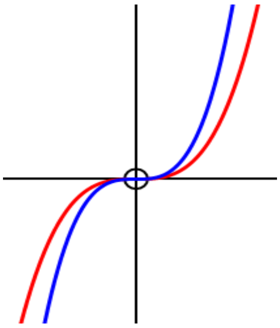
25



red: $y = \frac{1}{x^2}$

blue: $y = \frac{2}{x^2}$

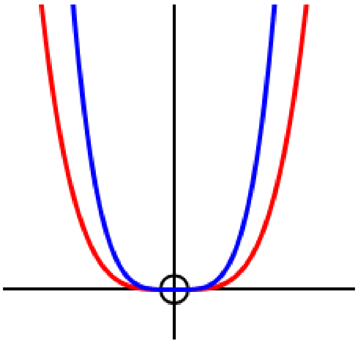
26



red: $y = x^3$

blue: $y = 2x^3$

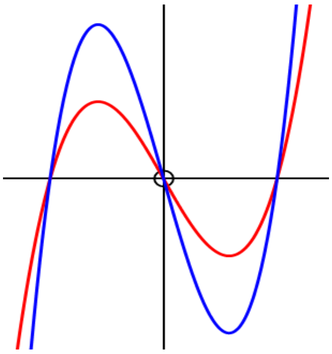
27



red: $y = x^4$

blue: $y = 3x^4$

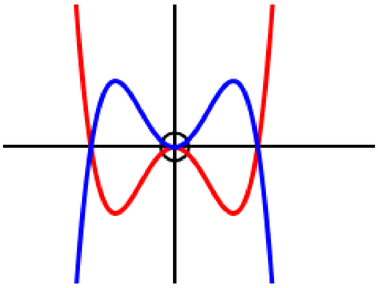
28



red: $y = x^3 - 4x$

blue: $y = 2x^3 - 8x$

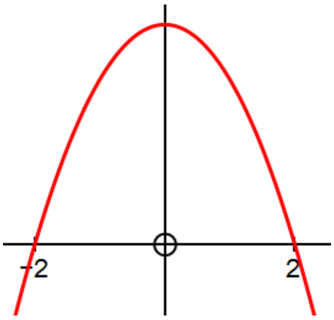
29



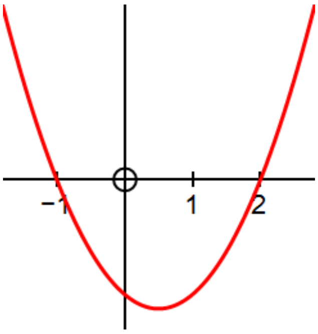
red: $y = x^4 - x^2$

blue: $y = -x^4 + x^2$

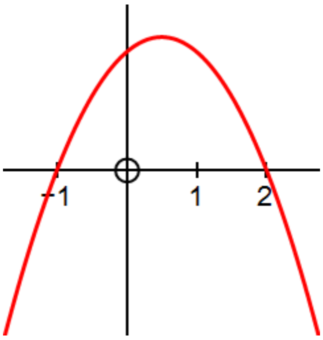
30 (a)



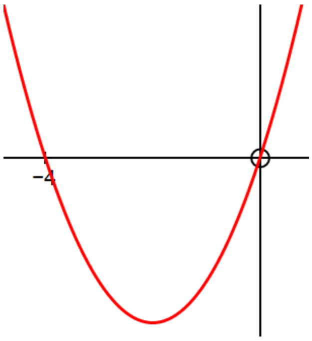
(b)



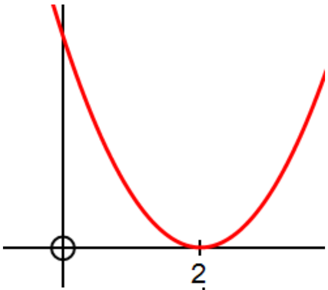
(c)



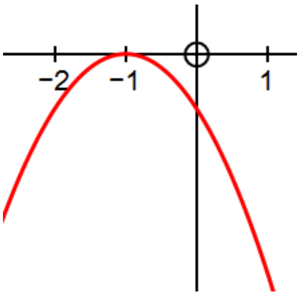
(d)



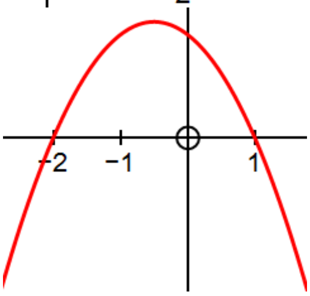
(e)



(f)



(g)



NOTE: in parts (b), (c) and (g) in particular, the maximum or minimum point is *not* on the *y*-axis.

2.19 Trigonometry

1	(a)	64.2, 115.8	(b)	53.1, 306.9	(c)	63.4, 243.4
	(d)	203.6, 336.4	(e)	120, 240	(f)	108.4, 288.4
2	(a)	64.2, 115.8	(b)	53.1, –53.1	(c)	63.4, –116.6
	(d)	–23.6, –156.4	(e)	120, –120	(f)	–71.5, 108.4

3. Solutions to the Practice Booklet Test

1) (a) $4x^2 + 4x - 3$ (b) $a^2 + 6a + 9$ (c) $10x^2 - 13x$

2) (a) $x(x-7)$ (b) $(y+8)(y-8)$ (c) $(2x-1)(x+3)$ (d) $(3t-5)(2t-1)$

3) (a) $\dfrac{x}{2y^2}$ (b) $\dfrac{10x+3}{6}$ (c) $\dfrac{6x-2}{(x-2)(x+3)}$

4) (a) h = 5 (b) $x = 0$ or $x = 8$ (c) p = -6 or p = 2

5) $x = 3$

6) (a) $x = 3, y = 4$ (b) $x = -4, y = 3$ and $= 3, y = -4$

7) (a) $x = \dfrac{v^2 - u^2}{2a}$ (b) $x = \sqrt{\dfrac{3V}{\pi h}}$ (c) $x = \dfrac{2-y}{y-1}$

8) $x = \dfrac{1 \pm \sqrt{21}}{10}$

9) (a) $x < 1.6$ (b) $x < -2.5$ (c) $x < -2$ or $x > 8$

10) (a) 19 (b) 32 (c) 11 (d) -23

11) a) $15\sqrt{6}$ b) $3\sqrt{6} + 3\sqrt{3}$ c) $\frac{5\sqrt{2}}{2}$ d) $\frac{3\sqrt{6}-2\sqrt{3}}{6}$ e) $\frac{5-4\sqrt{2}}{7}$

12) (a) $\theta = -348.5, -191.5, 11.5, 168.5$ (b) $\theta = \pm 110.5, \pm 249.5$

(c) $\theta = -274.5, -94.5, 85.5, 265.5$

NOW WRITE DOWN YOUR MARK OUT OF 50 and email a photo of this PRACTICE TEST to Mr Steinhobel (f.steinhobel@tuptonhall.org.uk).